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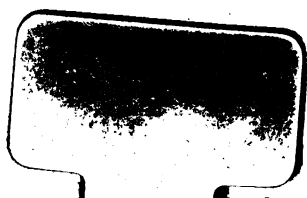




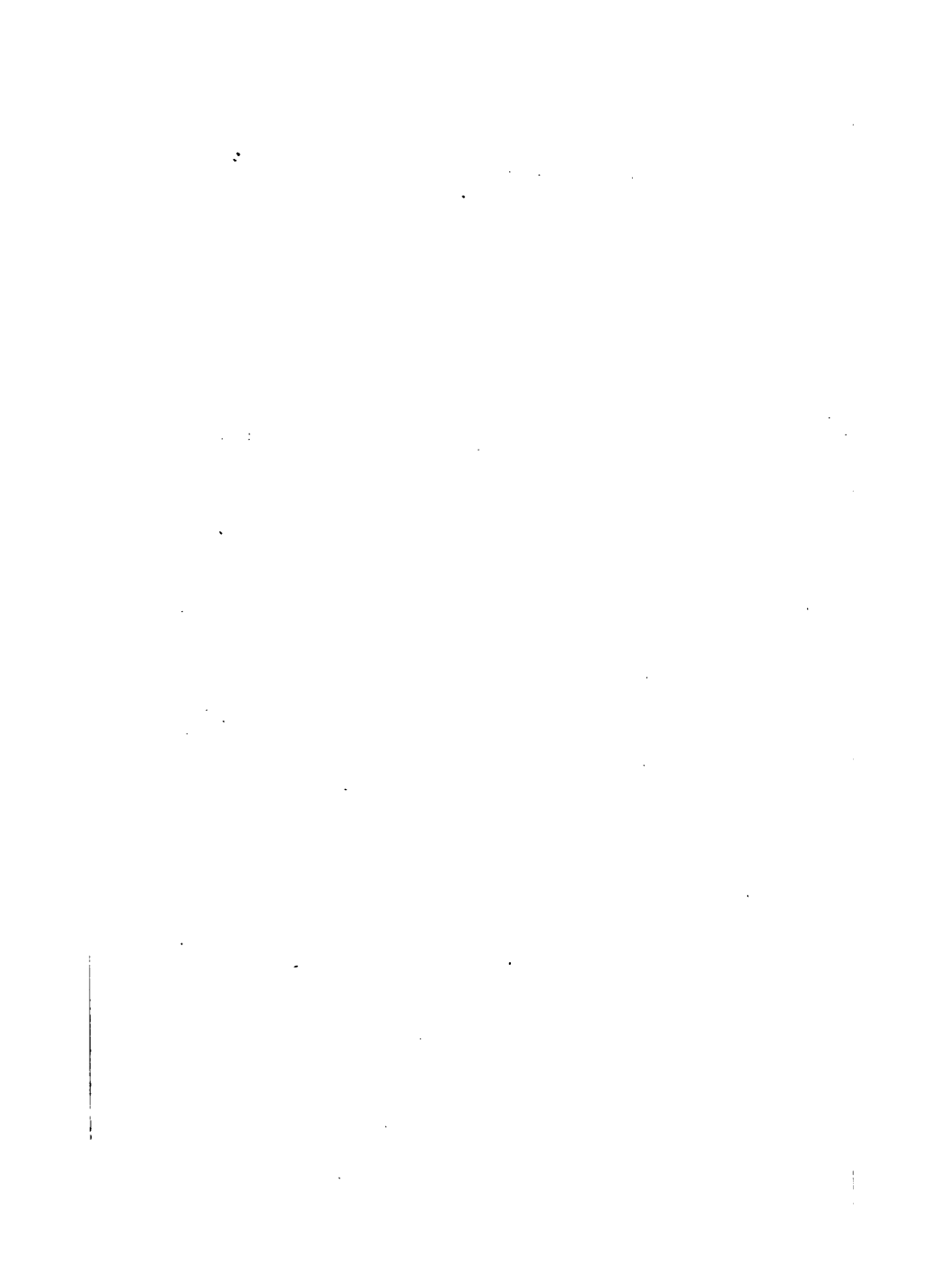












EUCLIDIAN GEOMETRY.



EUCLIDIAN GEOMETRY.

BY

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London :

MACMILLAN AND CO.

1874.

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183 . 4 . 72 .

Cambridge:

**PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.**

P R E F A C E.

BEFORE entering into a detailed account of the plan of the present work one characteristic may perhaps deserve especial prominence, namely that, while all those parts of the Elements of Euclid which are required by the Universities have been established and at the same time Problems separated from Theorems, both classified according to the nature of the subject, and demonstrations replaced by others less cumbrous, I have been careful in all cases to retain Euclid's order as far as is necessary to allow of the proofs given being substituted for those of Euclid in Examinations. How important this is must be at once apparent to all who are conversant with Examinations either at Schools or Universities.

One of the most generally acknowledged defects in the Elements of Euclid is the treatment of parallels; for a theorem is assumed as self-evident which certainly requires proof as much as those which are made to depend upon it. To avoid this difficulty a Lemma has been introduced which, while it rests on an axiomatic basis, brings out the property

of the parallelism (or in common language equidistance) of straight lines in the same plane which never meet. This is of great consequence, as the leading idea in connection with straight lines called parallel is that suggested by the literal meaning of the word: thus the conception of a parallelogram is not that of a figure whose opposite sides will never meet however far they may be produced in either direction, but rather that of a figure whose opposite sides are equidistant.

Again, Euclid's treatment of ratio and proportion is objected to as being only adapted to the grasp of the advanced geometrician; one of the great difficulties to be encountered by a student attempting to master Euclid's Fifth Book being that magnitudes in general are dealt with instead of certain specified magnitudes. Accordingly to supply the want due to the fact of this Book being invariably omitted, the necessary propositions contained in it have been established for those particular species of magnitude to which they are afterwards applied. The method given possesses the combined advantages of being rigorous, geometrical, simple and short. The terms duplicate ratio and compound ratio have been relegated to the Supplement, there placed for the benefit of those likely to be examined thereon, but ejected from the text, as not being generally appreciated by students, few of whom would estimate the relative magnitudes of plane similar rectilineal figures by contemplating their homologous sides and then calculating their duplicate ratio.

In order to make the Second Book, which treats of the relations existing between the rectangles under the segments

of a line divided in various ways, available for the extensive development of this part of Geometry of which it is capable, Euclid's second and eight following propositions should be shewn to be deducible from the first one. This has accordingly been done, the method being applicable to all propositions which are the geometrical interpretation of Algebraical identities of two dimensions.

At the latter part of the Third Book a section has been added on the equal subdivision of the circumference of a circle, thus preparing the way for a more general consideration of the subject of regular polygons.

In the Fourth Book the areas and perimeters of regular polygons inscribed in and circumscribed about a circle have been discussed, and the groundwork laid for a comparison not only of the areas but also of the circumferences of circles.

It has been deemed essential that Problems, though kept distinct from Theorems, should nevertheless occupy the attention of the beginner as soon as possible, and also that the necessity of his having to wade through a long book of Theorems only should be avoided. This has been effected by placing the Problems at the end of the various sections, which are never of any great length.

Euclid's mode of demonstration, in which the conclusion of each step is preceded by reasoning expressed with all the exactness of the minor premiss of a syllogism, of which some previous proposition is the major premiss, has been adopted as offering a good logical training, and also as being peculiarly adapted for teaching large classes, rendering it possible for the teacher to call first upon one

then upon another, and so on, to take up any link in the chain of argument.

The principle of superposition has been made use of whenever possible, as affording the most convincing proof of equality of geometrical magnitudes.

With a view to facilitating the student's progress it has been arranged that, as a rule, each proposition should have a page to itself, and that those of greater length should occupy pages facing each other: so also that propositions and the corresponding converse ones should be placed on opposite pages. Moreover, Definitions and Axioms, instead of being all at the beginning of the book, have been introduced when required.

The great objection entertained by those in authority at the Universities towards modern text-books on Geometry is grounded on the fact that enormous inconvenience would arise in conducting examinations with no recognized sequence of propositions. This objection I believe I have effectually dealt with by giving demonstrations which depend on propositions occupying a prior position not only in this work but also in Euclid.

How this and other points of general arrangement have been carried out will to some extent be seen by reference to the accompanying Tables and Scheme for Examinations. It will be readily seen that the scheme referred to offers fair rivalry between Euclid's proofs and others without necessarily displacing Euclid as a text-book. I think that there would thus be found no difficulty in conducting examinations under it, and this opinion is shared by Mr F. C. Wace, M.A., Fellow and Lecturer of St John's.

College, Cambridge, whose frequent experience as Moderator and Examiner for the Mathematical Tripos gives him a practical acquaintance with the difficulties which must accompany any such alteration, and to whom I am indebted for several corrections and suggestions in the revision of this work for the press.

In conclusion, it remains for me to acknowledge the debt of gratitude due to my former tutor, the Rev. J. G. Mould, B.D., late Fellow and Tutor of Corpus Christi College, Cambridge: to whom I submitted the work before placing it in the hands of the publishers last September, when he kindly offered many valuable criticisms, especially on my method of treating proportion. Since that time I have been pleased to find that the idea of establishing the various propositions on proportion for particular geometrical magnitudes has been approved of by some of the foremost mathematicians of the day.

CITY OF LONDON SCHOOL,

March, 1874.

TABLE I. *Shewing the Classification of Subjects.*

E.G.	Euclid.	E.G.	Euclid.	E.G.	Euclid.
PLANE GEOMETRY.		Quad. and Mult. Figs.		BOOK II.	
BOOK I.				Rectangles.	
Intr ^a ... II Cor.				I I, 2, 3	
St. lines, Ls and Ds.				2 4... 10	
I 4		26 34		3 12	
2 5		27 46 Cor.		4 13	
3 26a		28		A 14	
4 6		29		B 11	
5 8		30 32 Cor.		C IV. 10	
A 1		H VI. 9			
B 9		K 46		Sup. to Book II.	
C 10				a 4	
Right Ls.		Areas.		b 7	
6 15		31 35, 36		c 5, 6	
7 II		32 41, 37, 38		d 8	
8		33 39, 40		—	
9 13		34 43		BOOK III.	
10 14		35 47		Circles.	
11 12		36 48		Chords.	
12		L 42		I 2	
Inequalities.		M 44		2 I, 3	
13 16, 17		N 45		3 9	
14 18				4 14	
15 19		Sup. to Book I.		5 15	
16 20		a		A 25	
17 21		b		Tangents.	
18 24		c		6 16, 18, 19	
19 25		d		B 17	
20 22		e		C	
21 23		f		Contact of Os.	
22		g		7 5, 6	
23		h		8 10	
24		k		9 13	
25		l		10	
Parallels.		m II. 9, 10		11 11	
Lemma 31		n		12 12	
20 27, 28		p 2			
21 (29)		q 3			
22 (30)		r 7			
23 33		I. 21 29			
24 32		I. 22 30			
25 (36β)		I. 25 26β			
		I. 35 (47)			

TABLE I. *Shewing the Classification of Subjects.*

E.G.	Euclid.	E.G.	Euclid.	E.G.	Euclid.
	<i>∠s in ∘s.</i>		BOOK V.		Sup. to Book VI.
13.....	20		<i>Proportion.</i>	<i>a</i>	
14.....	21			<i>b</i>	
15.....	22			<i>c</i>	
16.....	31	Int.		<i>d</i>	7
17.....	32	1.....	(VI. 1)	<i>e</i>	
18.....	26—29	2.....	7	<i>f</i>	B
D	33	3.....	9β	<i>g</i>	C
E	34	4.....	11	<i>h</i>	D
F	30	5.....	VI. 16, 17	<i>i</i>	32
		6.....	16	<i>j</i>	21
		7.....	B	<i>k</i>	24
		8.....	A	<i>l</i>	26
<i>Subdivision of ∘s.</i>		9.....	14	<i>m</i>	14
G		10.....	12	<i>n</i>	15
H		11.....	12	<i>o</i>	I
K		12.....	22	<i>p</i>	33
L		13.....	22	<i>q</i>	19, 20
				<i>r</i>	22
Sup. to Book III.				<i>s</i>	23
<i>a</i>	4		BOOK VI.		
<i>b</i>	7, 8		<i>Comp. of st. lines.</i>		SOLID GEOMETRY.
<i>c</i>					<i>Planes.</i>
<i>d</i>	35	1.....	2	1.....	I
<i>e</i>	36, 37	2.....	3	2.....	2α
<i>f</i>		3.....	A	3.....	2β
<i>g</i>		4.....	4, 5, 6	4.....	7
<i>h</i>	23, 24	5.....	(III. 35, 36)	5.....	3
		6.....	8		
		A.....	10	<i>St. lines ⊥ to planes.</i>	
		B.....	12	6.....	4, 5
		C.....	11	7.....	6, 8
		D.....	13	8.....	11
		E.....	30	9.....	13α
				10.....	12
		<i>Comp. of rect. figs.</i>		11.....	13β
		7.....	19	<i>Parallel lines.</i>	
		8.....	20	12.....	9
		9.....	22	<i>Parallel planes.</i>	
		10.....	31	13.....	14
		F.....	18	14.....	16
		G.....	25	A.....	10, 15
				B.....	17
		<i>Comp. of ∘s.</i>		<i>Planes ⊥ to planes.</i>	
		11.....	XII. 1	15.....	18
		12.....	XII. 2, &c.	16.....	19
		<i>Comp. of ∠s.</i>		<i>Solid angles.</i>	
		13.....	(33)	17.....	20
				18.....	21

TABLE II. *For Comparison with Euclid.*

Euclid.	E.G.	Euclid.	E.G.	Euclid.	E.G.
Book I.		Book I.		Book III.	
1	A	34	26	1	2
2	<i>p</i>	35	3 ^I	2	1
3	<i>q</i>	36	3 ^I	3	2
4	i	37	3 ²	4	a
5	2	38	3 ²	5	7
6	4	39	33	6	7
7	r	40	33	7	b
8	5	41	3 ²	8	b
9	B	42	L	9	3
10	C	43	34	10	8
11	7	44	M	11	11
12	11	45	N	12	12
13	9	46	K	13	9
14	10	47	35	14	4
15	6	48	36	15	5
16	13			16	6
17	13			17	B
18	14			18	6
19	15			19	6
20	16			20	13
21	17			21	14
22	D			22	15
23	E			23	<i>h</i>
24	18			24	<i>h</i>
25	19			25	A
26 ... 3 and Sup.(25)				26	18
27	20			27	18
28	20			28	18
29 Sup.(21)				29	18
30 Sup.(22)				30	F
31 Lemma.				31	16
32	24, 30			32	17
33	23			33	D
				34	E
				35	d (VI. 5)
				36	c (VI. 5)
				37	c (VI. 5)

TABLE II. *For Comparison with Euclid.*

Euclid.	E.G.	Euclid.	E.G.	Euclid.	E.G.
Book IV.		Book VI.		Book XI. SOL. GEO.	
1	1	1	<i>o</i> (v. 1)	1	1
2	2	2	1	2	2, 3
3	3	3	2	3	5
4	5	A	3	4	6
5	4	4	4	5	6
6	6	5	4	6	7
7	7	6	4	7	4
8	9	7	<i>d</i>	8	7
9	8	8	6	9	12
10	II. C	9	1. H	10	A
11 ... 10 and III. K		10	A	11	8
12	11	11	C	12	10
13	13	12	B	13	9, 11
14	12	13	D	14	13
15 ... 10 and III. H		14	<i>m</i>	15	A
16 ... 10 and III. L		15	<i>n</i>	16	14
—		16	v. 5	17	B
Book V.		17	v. 5	18	15
A	8	18	F	19	16
B	7	19	<i>q</i> 7	20	17
7	2	20	<i>q</i> 8	21	18
9 β	3	21	<i>j</i>	—	
II	4	22	<i>r</i> 9	Book XII. Book VI.	
12	10, 11	23	<i>s</i>	1	II
14	9	24	<i>k</i>	2	12
16	6	25	G		
22	12, 13	26	<i>l</i>		
		30	E		
		31	10		
		32	<i>i</i>		
		33	<i>p</i> 13		
		B	<i>f</i>		
		C	<i>g</i>		
		D	<i>h</i>		

SCHEME FOR EXAMINATION ON GEOMETRY.

*Extent of Examination.**Proofs allowed.*

All those portions of Euclid required by the Universities. { Proofs of Euclid or others depending on previous propositions of Euclid.

Also those Propositions on Proportion required for Euclid VI, XI, XII, established for the particular Geometrical magnitudes to which they are applied. { In the proofs sent up (if not Euclid's Book V.) the Definitions and Enunciations of previous Theorems on Proportion made use of to be stated.

With regard to *Euc. I.* { Any proofs allowed, the Axiom or Lemma on which they depend being stated.
29, 30.

With regard to *Euc. VI.* { Any proofs allowed, the Definition or test of Proportion being stated.
I. 33.

N.B. When the Examiner wishes for other proofs than the above, the necessary restrictions and allowable assumptions may be given in the question.

PRELIMINARY NOTICES.

The following symbols will be used simply in lieu of the words to which they are here prefixed.

$\angle$ angle.	$+$ together with.
Δ triangle.	$=$ equal to.
$\square$ parallelogram.	$>$ greater than.
$\odot$ circle.	$<$ less than.
$\bigcirc$ ce circumference.	$\therefore$ because.
$\perp$ perpendicular (to).	$\therefore$ therefore.
$\parallel$ parallel.	$:$ is to.

The object aimed at is merely to place before the student the various steps of an argument in a more succinct form by curtailing the verbiage, thus enabling him to take in at a glance the outline of a proof. It must however be borne in mind that the words themselves must be substituted in any examinations in which symbols are objected to.

It will be assumed as self-evident that :—

Things which are equal to the same thing are equal to one another.

The whole is equal to the sum of its parts.

If equals be added to equals the wholes are equal.

If equals be taken from equals the remainders are equal.

If equals be added to unequals the wholes are unequal.

If equals be taken from unequals the remainders are unequal.

Things which are double of equal things are equal to one another.

Things which are halves of equal things are equal to one another.

Equimultiples of equal things are equal.

Things, of which the equimultiples are equal, are themselves equal.

If one magnitude be greater than another any multiple of the former is greater than the same multiple of the latter.

If a multiple of one magnitude be greater than the same multiple of another then the former magnitude is greater than the latter.

ERRATA.

P. 12, l. 7, for equal circles read $\odot$ s having equal radii.

„ 24, l. 10 and p. 27, l. 20, 22, for has been read is.

„ 38, l. 4, for FE read FG.

„ 71, „ 13, 16. For definitions, see p. 86.

„ 107, „ 6, for divides read divide.

„ 129, „ 1, add if possible, let them bisect each other;

„ 131, „ 13, omit to it.

„ 201. In the diagram the diagonal of $\square$ FB is AC.

PLANE GEOMETRY.

BOOK I.

INTRODUCTION.

A point has no magnitude. It has position.

A line has length but not breadth.

The extremities of lines are points.

A straight line is one which lies evenly between its extremities.

It is taken for granted :

That a straight line may be drawn from any one point to any other point ;

And also that a terminated straight line may be produced to any length in a straight line.

A superficies has length and breadth but not thickness. The extremities of superficies are lines.

A plane is a superficies in which any two points being taken the straight line joining them lies wholly in that superficies.

An angle is formed by two lines drawn from a point.

The two lines are called the **arms**, and the point the **vertex** of the angle.

If the arms are in one plane the angle is called a **plane angle**, and if they are straight lines a **plane rectilineal angle**.

Whenever the term angle is used a plane rectilineal angle is meant unless otherwise expressed.

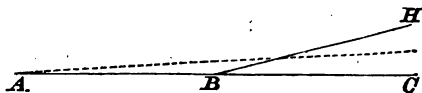
The size of an angle does not depend on the length of its arms but on their direction, and is said to be greater or less according as the difference of direction of its arms is greater or less.

It is evident that

Two straight lines cannot enclose a space.

Hence it follows that

Two straight lines cannot have a common segment.



For if two straight lines ABC , ABH could have a common segment AB ;

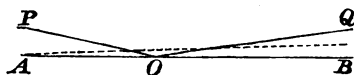
then the straight line ABC might be turned about its extremity A , towards the side on which BH is, so as to cut BH ;

and thus two straight lines would enclose a space, which is impossible.

Hence it follows that

If two straight lines pass through the same point they will coincide entirely or cut one another.

For if not, if possible let them fall otherwise as AOB , POQ having a common point O .



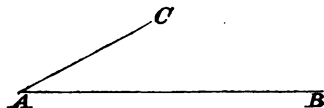
Then AOB might be turned about one extremity A towards the side on which OQ is, so as to cut OQ and also OP .

Thus two straight lines would enclose a space, which is impossible.

It is evident that

Magnitudes which may be made to coincide, i.e. exactly fill the same space, are equal to one another.

If two angles can be so placed that their vertices coincide and the arms of one fall along the arms of the other, those angles are said to be equal to one another,



If the angle BAC were taken up, reversed, and then applied to its former position, so that A fell on its former position and AC along that of AB ; then would AB fall along that of AC .

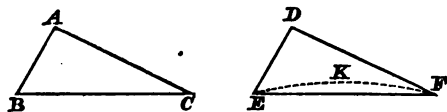
STRAIGHT LINES, ANGLES AND TRIANGLES.

DEFINITION.

A triangle is a plane figure contained by three straight lines.

PROPOSITION I.

If two sides and the included angle of one triangle be respectively equal to two sides and the included angle of another triangle, the triangles shall be equal in all respects.



In the Δs ABC , DEF ,
let the two sides AB , AC be respectively equal to DE , DF ,
and $\angle BAC = \angle EDF$;
then shall the Δs ABC , DEF be equal in all respects.

For since the $\angle BAC$ is = the $\angle EDF$,
the ΔABC may be applied to the ΔDEF so that A
may fall on D , and the straight lines AB , AC along DE , DF ;
then B will fall on E , $\because AB$ is = DE ,
and C will fall on F , $\because AC$ is = DF ;
 $\therefore BC$ will coincide with EF .

For if it fell otherwise, as EKF , then two straight lines
would enclose a space, which is impossible;

$\therefore BC$ is = EF .

Also the ΔABC will coincide with and is $\therefore = \Delta DEF$,

$\angle ABC$ will coincide with and is $\therefore = \angle DEF$;

so

$\angle ACB$ is = $\angle DFE$.

Hence the Δs ABC , DEF are equal in all respects.

DEFINITION.

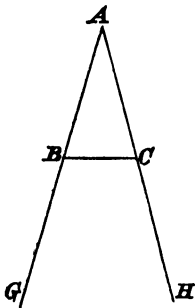
An isosceles triangle is one which has two of its sides equal.

PROPOSITION II.

The angles at the base of an isosceles triangle are equal; and if the equal sides be produced, the angles on the other side of the base shall be equal.

Let ABC be an isosceles Δ , having the side $AB =$ the side AC , then shall the $\angle ABC$ be $=$ the $\angle ACB$; also let the equal sides AB, AC be produced to G, H ;

then shall $\angle GBC$ be $= \angle HCB$.



For if the figure were taken up, reversed, and then applied to its former position so that A might fall on its former position and ABG along that of ACH ;

then would ACH fall along that of ABG ;

also B would fall on that of C , $\because AB$ is $= AC$,

and C on that of B for the same reason,

$\therefore BC$ would coincide with its former position.

Thus AB, BC would fall on the former positions of AC, CB ;

$\therefore \angle ABC$ is $= \angle ACB$.

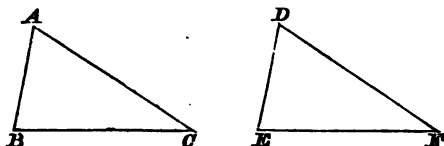
Also BG, BC would fall along the former positions of CH, CB ;

$\therefore \angle GBC$ is $= \angle HCB$.

6 *STRAIGHT LINES, ANGLES AND TRIANGLES.*

PROPOSITION III.

If two angles and the adjacent side of one triangle are respectively equal to two angles and the adjacent side of another triangle, then shall the triangles be equal in all respects.



In the $\triangle s$ ABC , DEF , let the $\angle s$ ABC , ACB and the adjacent side BC be respectively equal to the $\angle s$ DEF , DFE and the adjacent side EF .

Then shall the $\triangle s$ ABC , DEF be equal in all respects.

For if $\triangle ABC$ be applied to $\triangle DEF$ so that BC may fall on EF ;

then will BA fall along ED , $\because \angle ABC$ is $= \angle DEF$,

and CA will fall along FD , $\because \angle ACB$ is $= \angle DFE$;

$\therefore$ the point A will fall on the point D ,

$\therefore$ the triangle ABC will coincide with the $\triangle DEF$, and is

$\therefore$ equal to it in all respects.

PROPOSITION IV.

If two angles of a triangle are equal to one another, the sides opposite to those angles are equal.



Let ABC be a triangle, having
the $\angle ABC = \text{the } \angle ACB$,
then shall the side AC be = the side AB .

For if the $\triangle ABC$ be taken up, reversed, and applied to the former position of BC and on the same side of it as before, so that C may fall on the former position of B and B on that of C ;

then will CA fall along that of BA , $\because \angle ABC$ is = $\angle ACB$,
and BA along that of CA for the same reason;

$\therefore A$ will fall on its former position,

and $\therefore AC$ will coincide exactly with that of AB ,

$\therefore AC$ is = AB .

8 STRAIGHT LINES, ANGLES AND TRIANGLES.

PROPOSITION V.

If the three sides of one triangle are respectively equal to the three sides of another, the triangles shall be equal in all respects.

Let the sides AB, BC, CA of the $\triangle ABC$ be respectively equal to the sides DE, EF, FD of the $\triangle DEF$.

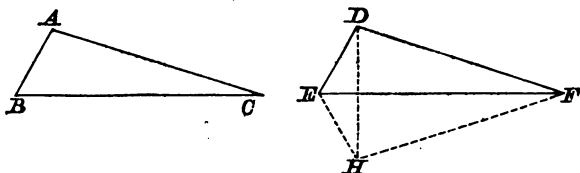
Then shall the $\triangle s\ ABC, DEF$ be equal in all respects.

Suppose the $\triangle ABC$ taken up, reversed, and placed so that BC may fall on EF ,

and A on the opposite side of EF to that on which D is, as H .

Join HD .

(i) If HD cuts EF .



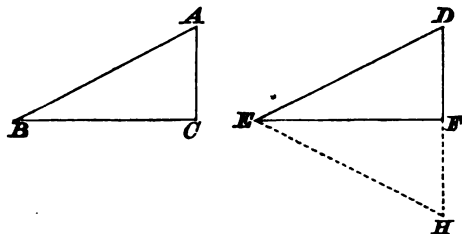
$\therefore EH$ is $= ED$, $\therefore \angle EHD$ is $= \angle EDH$; (I. 2)

and $\therefore FH$ is $= FD$, $\therefore \angle FHD$ is $= \angle FDH$; (I. 2)

$\therefore$ the whole $\angle EHF$ is $=$ the whole $\angle EDF$,

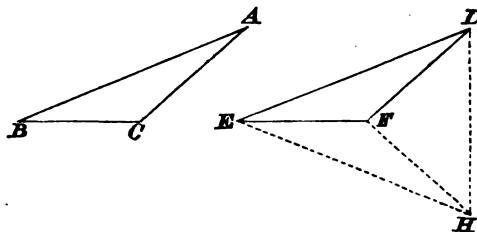
and $\therefore \angle BAC$ is $= \angle EDF$.

(ii) If HD passes through one extremity of EF as F .



$\therefore EH$ is $= ED$, $\therefore \angle EHD$ is $= \angle EDH$; (I. 2)
and $\therefore \angle BAC$ is $= \angle EDF$.

(iii) If HD does not meet EF .



$\therefore EH$ is $= ED$, $\therefore \angle EHD$ is $= \angle EDH$; (I. 2)
and $\therefore FH$ is $= FD$, $\therefore \angle FHD$ is $= \angle FDH$; (I. 2)
 $\therefore \angle EHF$ is $= \angle EDF$,
and $\therefore \angle BAC$ is $= \angle EDF$.

Thus in any case the

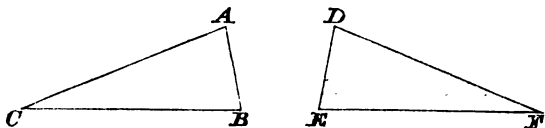
$$\angle BAC \text{ is } = \angle EDF,$$

also BA , AC are respectively $= ED$, DF ;

$\therefore \triangle s ABC, DEF$ are equal in all respects. (Prop. 1.)

10 *STRAIGHT LINES, ANGLES AND TRIANGLES.*

Note.—Propositions I, III, and V hold if the triangles are situated thus :



having the sides AB , BC , CA corresponding to DE , EF , FD .

In propositions I and III it will be necessary, after taking up $\triangle ABC$, to reverse it before applying it to $\triangle DEF$.

In proposition V, the $\triangle ABC$ must be applied without being reversed.

The demonstrations will be the same as before.

DEFINITIONS.

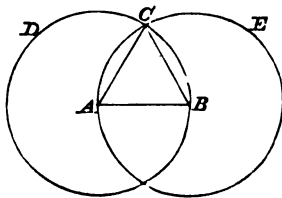
An equilateral triangle is one which has all its sides equal.

A circle is a plane figure contained by one line called the **circumference**, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal. That point is called the **centre** of the circle, and those straight lines **radii**.

It is taken for granted that a circle can be described having any centre and radius equal to any given straight line.

PROBLEM A.

Describe an equilateral triangle upon a given straight line.



Let AB be the given straight line.

It is required to describe an equilateral triangle upon AB .

With A as centre, and radius AB , describe the circle BCD , and with B as centre and radius BA describe the circle ACE .

Let these circles intersect in C ; join CA , CB .

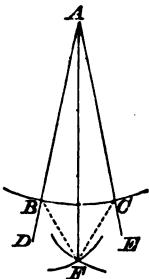
Then ABC shall be the equilateral triangle required.

$\therefore A$ is the centre of the $\odot BCD$, $\therefore AB$ is $= AC$,
 and $\therefore B$ is the centre of the $\odot ACE$, $\therefore AB$ is $= BC$,
 $\therefore AC$ is $= BC$;
 $\therefore ABC$ is an equilateral triangle, and it has been described
 upon AB . Q. E. F.

12 STRAIGHT LINES, ANGLES AND TRIANGLES.

PROBLEM B.

To bisect a given angle.



Let DAE be the given angle.

It is required to bisect it.

With centre A describe a circle cutting AD , AE in the points B and C .

With centres B and C describe equal circles cutting one another in F .

Join AF ; then $\angle DAE$ shall be bisected by AF .

Join BF , CF .

Then $\therefore$ in the $\triangle s$ ABF , ACF ,

BA is $= CA$, $BF = CF$, and AF is common ;

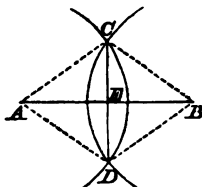
$\therefore$ the $\triangle s$ ABF , ACF are equal in all respects, (I. 5)

and $\therefore$ the $\angle BAF$ is $=$ the $\angle CAF$,

$\therefore \angle DAE$ has been bisected by AF . Q. E. F.

PROBLEM C.

To bisect a given straight line.



Let AB be the given straight line.

It is required to bisect it.

With centres A and B describe two circles having equal radii intersecting in C, D .

Join CD cutting AB in E ;

then AB shall be bisected in E .

Join AC, CB, BD, DA .

Then $\therefore$ in the $\triangle s ACD, BCD$,

AC, CD, DA are respectively $= BC, CD, DB$,

$\therefore \triangle s ACD, BCD$ are equal in all respects,

and $\therefore \angle ACD$ is $= \angle BCD$. (I. 5)

Hence in the $\triangle s ACE, BCE$,

AC, CE and the included $\angle ACE$ are respectively $= BC, CE$ and the included $\angle BCE$;

$\therefore \triangle s ACE, BCE$ are equal in all respects,

and $\therefore AE$ is $= BE$; (I. 1)

$\therefore AB$ has been bisected in E . Q. E. F.

RIGHT ANGLES.

DEFINITION.

When one straight line, standing on another straight line, makes the adjacent angles equal to one another, each of them is called a **right angle**; and the straight line which stands upon the other is said to be **perpendicular** to it.

PROPOSITION VI.

If two straight lines cut one another the opposite angles shall be equal.



Let two intersecting straight lines form the four angles A, B, H, K .

Then shall the opposite angles A, B be = one another.

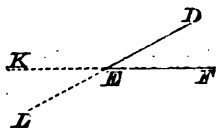
For if the figure were taken up, reversed, and placed so that each of the arms of H might fall along the former position of the other arm; then each of these lines produced would fall along the former position of the other.

Thus the arms of A would fall along the former positions of the arms of B ;

$\therefore$ the angles A, B are equal to one another.

Similarly the angles H, K may be proved equal.

COR. *If DEF be any angle, the adjacent $\angle DEK$ formed by producing one of its arms FE is = the adjacent $\angle FEL$ formed by producing the other arm.*



For they are opposite angles.

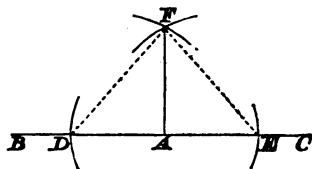
$\therefore$ if an angle = one of its adjacent angles it is also = the other,

$\therefore$ if one straight line is $\perp$ to another, the latter is $\perp$ to the former.

Thus *each* of the arms of a right $\angle$ is $\perp$ to the other.

PROPOSITION VII.

It is always possible to draw a straight line perpendicular to a given straight line from a given point in the same.



Let BC be the given straight line,
and A the given point in it.

Then from A a straight line can be drawn $\perp BC$.

From AB, AC cut off equal parts AD, AE .

With D and E as centres, describe equal $\bigcirc$ s intersecting in F .

Join AF .

Then AF shall be $\perp$ to BC .

Join DF, EF .

$\therefore DA$ is $= EA$, $DF = EF$, and AF common to the two

$\triangle$ s DAF, EAF ,

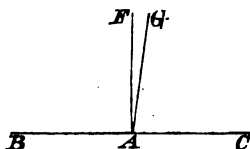
$\therefore$ these $\triangle$ s are equal in all respects; (1. 5)

$\therefore \angle DAF$ is $= \angle EAF$,

$\therefore AF$ is $\perp$ to BC . (Def.)

PROPOSITION VIII.

From a given point in a given straight line there cannot be drawn more than one straight line perpendicular to it.



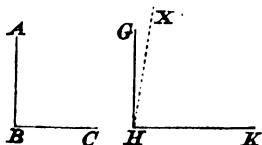
For if possible

let AF and AG be each of them $\perp$ to BC .

$\therefore \angle s$ FAB, FAC are equal to one another,
and also $\angle s$ GAB, GAC are equal to one another,
which is impossible.

$\therefore$ from a given point in a given straight line there cannot be drawn more than one straight line $\perp$ to it.

COR. *All right angles are equal to one another.*



Let ABC, GHK be two right angles,
and let the $\angle ABC$ be applied to the $\angle GHK$,
so that B may fall on H , and BC along HK ,
then will BA fall along HG .

For if it fell in any other direction, as HX , then from H there would be drawn two straight lines HG, HX , $\perp$ to HK , which is impossible;

$\therefore BA$ must fall along HG ,

and

$\therefore \angle ABC$ is $= \angle GHK$.

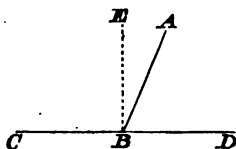
C. G.

PROPOSITION IX.

The angles which one straight line makes with another upon one side of it are either two right angles or are together equal to two right angles.

Let the straight line AB make with CD upon one side of it the $\angle$ s ABC , ABD .

These shall either be two right $\angle$ s;
or shall together be = two right $\angle$ s.



If the $\angle ABC$ is = the $\angle ABD$;
then each of them is a right $\angle$.

But if not,

through B draw $BE \perp$ to CD ;

then the $\angle ABC$ is = the $\angle$ s CBE and EBA together;

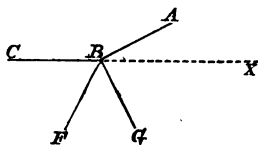
$\therefore$ the $\angle$ s ABC and ABD are together = the $\angle$ s CBE , EBA and ABD ;

of which CBE is a right $\angle$,

and EBA , ABD make up a right $\angle$;

$\therefore \angle$ s ABC , ABD are together = two right $\angle$ s.

COR. If a number of straight lines BA , BC , BF , BE meeting in the point B form angles ABC , CBF , FBG , GBA which fill up the space about B , then these $\angle$ s are together = 4 right $\angle$ s.

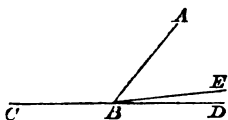


For produce CB to X .

Then all the $\angle$ s ABC , CBF , FBG , GBA are equal to the $\angle$ s on one side of CBX together with the $\angle$ s on the other side of CBX , and are $\therefore$ = four right $\angle$ s.

PROPOSITION X.

If two straight lines, drawn from one extremity of a straight line on opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.



Let BC , BD drawn from B one extremity of AB make the adjacent $\angle$ s ABC , ABD together = two right angles.

Then shall BD be in the same straight line with BC .

For if not, if possible, let BE be in the same straight line with BC ;

$\therefore \angle$ s ABC , ABE are together = two right angles; (I. 9)

and are $\therefore = \angle$ s ABC , ABD ; (hyp.)

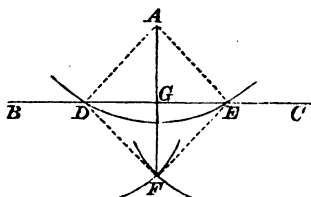
$\therefore \angle ABE$ is = $\angle ABD$, which is impossible.

$\therefore$ no other straight line than BD is in the same straight line with BC ;

$\therefore BD$ is in the same straight line with BC .

PROPOSITION XI.

It is always possible to draw a perpendicular to a given straight line of an unlimited length from a given point without it.



Let A be any point without the straight line BC of unlimited length.

Then from A a straight line can be drawn $\perp$ to BC .

With centre A describe a $\odot$ cutting BC in D and E .

With centres D and E describe equal $\odot$ s intersecting in F .

Join AF , cutting BC in G .

Then shall AG be $\perp$ to BC .

Join AD , AE , FD , FE .

Then $\because DA$, AF , FD are respectively $= EA$, AF , FE ;

$\therefore \triangle s DAF$, EAF are equal in all respects,

and $\therefore \angle DAF$ is $= \angle EAF$. (I. 5)

Again, $\because DA$, AG and the included $\angle DAG$ are respectively $= EA$, AG , and $\angle EAG$;

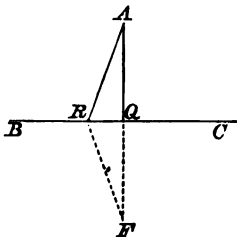
$\therefore$ the $\triangle s DAG$, EAG are equal in all respects;

$\therefore \angle DGA$ is $= \angle EGA$; (I. 1)

$\therefore AG$ is $\perp$ to BC .

PROPOSITION XII.

There cannot be drawn more than one perpendicular to a given straight line from a given point without it.



Let A be the given point, and BC the given straight line ;
and let AQ be drawn from $A \perp$ to BC . (I. 10)

Then no other straight line besides AQ can be drawn
from $A \perp$ to BC .

Draw any other straight line AR from A to BC .

Produce AQ , and make the produced part $QF = AQ$.

Join FR .

Then $\because \angle AQR$ is a right $\angle$, $\therefore \angle FQR$ is a right $\angle$. (I. 9)

Hence in the Δ s AQR , FQR ,

AQ , QR and $\angle AQR$ are respectively equal to FQ , QR ,
and $\angle FQR$;

$$\therefore \angle ARQ = \angle FRQ. \quad (I. 1)$$

$\therefore$ if ARQ were a right angle,

the $\angle$ s ARQ , FRQ would be together = two right angles,

and $\therefore ARF$ would be a straight line ; (I. 10)

and thus two straight lines would inclose a space, which
is impossible.

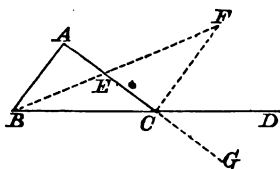
$\therefore AR$ is not $\perp$ to BC .

Similarly it may be proved that no other straight line
besides AQ , drawn from A to BC , is $\perp$ to it.

INEQUALITIES.

PROPOSITION XIII.

If a side of a triangle be produced the exterior angle is greater than either of the interior and opposite angles.



Let the side BC of $\triangle ABC$ be produced to D .
Then shall the $\angle ACD$ be $>$ either of the $\angle$ s BAC, ABC .

Bisect AC in E ; join BE and produce BE to F ,
making $EF = BE$.

Join CF .

Then, $\because AE, EB$ are respectively $= CE, EF$,
and $\angle AEB$ is $= \angle CEF$; (I. 6)

$\therefore$ the $\angle EAB$ is $=$ the $\angle ECF$. (I. 1)

But the $\angle ACD$ is $>$ the $\angle ECF$;

$\therefore \angle ACD$ is $>$ the $\angle EAB$; i.e. $\angle CAB$.

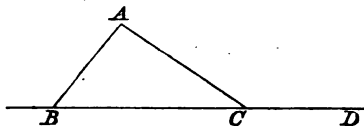
Similarly it may be shewn, if AC be produced to G ,
that $\angle BCG$ is $>$ the $\angle ABC$.

But $\angle ACD$ is $= \angle BCG$; (I. 6)

$\therefore \angle ACD$ is $>$ the $\angle ABC$.

Hence it follows that

Any two angles of a triangle are together less than two right angles.



Let ABC be a $\triangle$.

Then shall any two of its angles, as ABC and ACB , be together $<$ two right angles.

Produce BC to D .

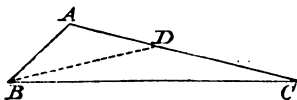
Then $\angle ABC$ is $<$ the exterior $\angle ACD$; (I. 13)

$\therefore \angle s\ ABC, ACB$ are together $<$ the $\angle s\ ACD, ACB$,

and are $\therefore <$ than two right angles. (I. 9)

PROPOSITION XIV.

The greater side of every triangle has the greater angle opposite to it.



Let ABC be a $\triangle$ having the side AC greater than the side AB .

Then shall the $\angle ABC$ be greater than the $\angle ACB$.

From AC cut off $AD = AB$,
and join BD .

Then $\because AB$ is $= AD$; $\therefore \angle ABD$ is $= \angle ADB$. (I. 2)

And $\because$ the side CD of $\triangle BCD$ has been produced to A ,

$\therefore \angle ADB$ is greater than $\angle DCB$; (I. 13)

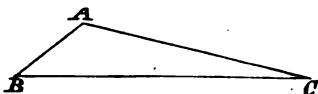
$\therefore$ also $\angle ABD$ is greater than $\angle DCB$.

Much more then is $\angle ABC$ greater than $\angle DCB$,

i.e. $\angle ACB$.

PROPOSITION XV.

The greater angle of every triangle has the greater side opposite to it.



Let ABC be a $\triangle$ having the $\angle ABC >$ the $\angle ACB$,
then shall the side AC be $>$ the side AB .

For if not, AC must either be $= AB$ or $< AB$.

AC is not $= AB$, for, then would $\angle ABC$ be $= \angle ACB$,
but it is not. (I. 2)

And AC is not $< AB$, for then would $\angle ABC$ be $<$
 $\angle ACB$, but it is not. (I. 14)

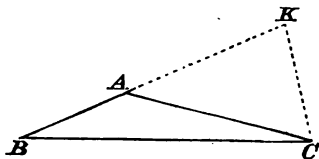
$\therefore AC$ is $> AB$.

PROPOSITION XVI.

Any two sides of a triangle are together greater than the third.

Let ABC be a Δ .

Then shall any two of its sides, as AB , AC , be together $> BC$.



Produce BA to K , making $AK = AC$, and join KC .

Then $\because AK$ is $= AC$; $\therefore \angle ACK$ is $= \angle AKC$; (I. 2)

but $\angle BCK$ is $> \angle ACK$;

$\therefore \angle BCK$ is $> \angle AKC$, i.e. $\angle BKC$;

$\therefore BK$ is $> BC$; (I. 15)

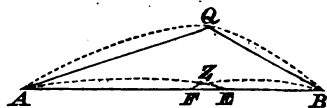
but BK is $= BA$ and AC together;

$\therefore BA, AC$ are together $> BC$.

COR. *Let two points A, B be joined by a path which is as short as any which can be drawn between them. Then every point in this path shall lie in the straight line AB .*

For if not, if possible, let a point Q in the path be without AB .

Join AQ, BQ .



From AB cut off $AE = AQ$, and $BF = BQ$.

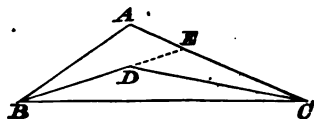
Then to AE, BF may be applied the portions of the path AQ, BQ , meeting in some point Z , and thus forming a path AZB between A and B shorter than the former, which is impossible.

Hence it follows that

the shortest distance between two points is a straight line.

PROPOSITION XVII.

If from the ends of one of the sides of a triangle there be drawn two straight lines to a point within the triangle: these shall be less than the other two sides of the triangle but shall contain a greater angle.



From the ends B , C of the side BC of the $\triangle ABC$ let the two straight lines BD , CD be drawn to a point D within the $\triangle$;

then shall BD , DC together be less than BA and AC ,
but the $\angle BDC$ shall be greater than $\angle BAC$.

Produce BD to meet the side AC in E .

Then BA and AE are together greater than BE ;
add to each EC ; (I. 16)

$\therefore BA$ and AC are together greater than BE and EC .

Again, DE and EC are together greater than DC ;
add to each BD ; (I. 16)

$\therefore BE$ and EC are together greater than BD and DC .

Much more then are BA and AC greater than BD and DC .

Also, since ED of $\triangle CED$ has been produced to B ;

$\therefore \angle BDC$ is greater than $\angle DEC$; (I. 13)

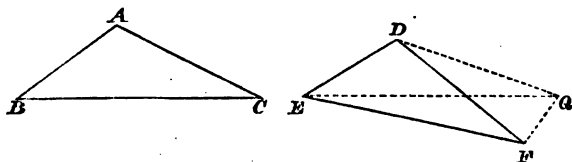
and $\therefore AE$ of $\triangle ABE$ has been produced to C ;

$\therefore \angle BEC$ is greater than $\angle BAE$.

Much more then is $\angle BDC$ greater than $\angle BAE$,
i.e. $\angle BAC$.

PROPOSITION XVIII.

If two triangles have two sides of the one equal to two sides of the other; but the angle contained by the two sides of one of them greater than the angle contained by the two sides of the other; then shall the base of that which has the greater angle be greater than the base of the other.



In the two Δ s ABC, DEF ,
 let AB, AC be respectively equal to DE, DF ,
 but the $\angle BAC$ be $>$ the $\angle EDF$,
 then shall BC be $> EF$.

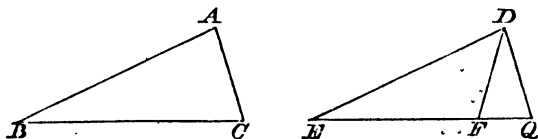
Place the ΔABC so that AB shall fall on DE , the point C falling on Q on the same side of DE as F .

1st. Let EQ cut DF : join FQ .

Then $\because DF$ is $= DQ$, $\therefore \angle DFQ$ is $= \angle DQF$; (I. 2)
 and is $> \angle EQF$.

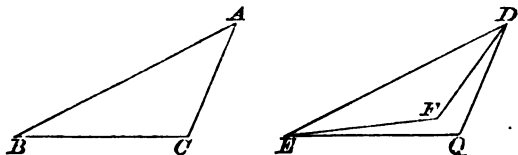
Much more then is $\angle EFQ > \angle EQF$;
 $\therefore EQ$ is $> EF$; (I. 15)
 i.e. BC is $> EF$.

2nd. Let EQ pass through F .



Then it is evident that EQ is $> EF$.
 i.e. BC is $> EF$.

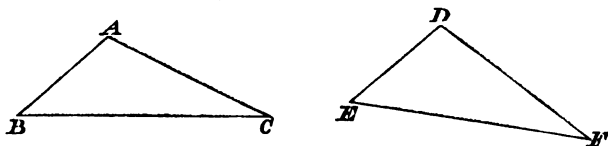
3rd. Let EQ not meet DF .



Then $\because EQ, QD$ are together $> EF, FD$, (I. 17)
 and QD is $= FD$;
 $\therefore EQ$ is $> EF$, i.e. BC is $> EF$.

PROPOSITION XIX.

If two triangles have two sides of the one equal to two sides of the other, but the base of one greater than the base of the other; then the angle contained by the sides of that triangle which has the greater base shall be greater than the angle contained by the sides equal to them of the other.



Let ABC, DEF be two Δ s having the sides BA, AC respectively equal to the sides ED, DF ;

but the base $BC >$ than the base EF .

Then shall the $\angle BAC$ be $>$ the $\angle EDF$

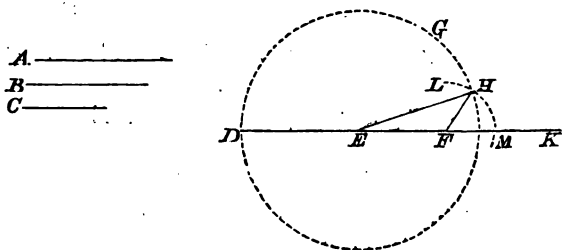
For if the $\angle BAC$ were $<$ the $\angle EDF$,
 then would the base BC be $<$ the base EF , (I. 18),
 but it is not;

and if the $\angle BAC$ were $=$ the $\angle EDF$,
 then would the base BC be $=$ the base EF , (I. 4),
 but it is not;

$\therefore$ the $\angle BAC$ must be $>$ the $\angle EDF$.

PROBLEM D.

Describe a triangle having its sides equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be the three given straight lines, any two of them being together greater than the third.

It is required to describe a $\triangle$ having its sides equal to A, B, C respectively.

Take a straight line DK unlimited towards K ; cut off DE, EF, FM equal to A, B, C respectively.

With E as a centre and radius ED describe the circle DGH ;

and with F as a centre and radius FM describe the circle LHM .

Join EH, FH .

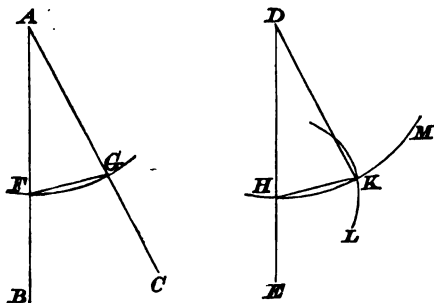
Then EFH is the $\triangle$ required.

For its sides HE, EF, FH are respectively equal to DE, EF, FM ;

and therefore to A, B, C .

PROBLEM E.

At a point in a given straight line to make an angle equal to a given angle.



Let BAC be the given angle.

It is required at the point D in DE to make an $\angle = \angle BAC$.

With centre A describe a circle cutting AB , AC in F and G ; and with centre D describe an equal circle HKM cutting DE in H .

Join FG ; with centre H and radius $= FG$ describe a circle cutting the circle HKM in K .

Join DK ; then EDK is the $\angle$ required.

Join HK :

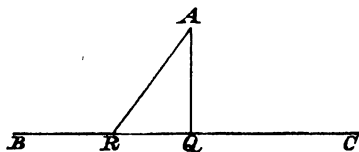
Then, $\because HD, DK, KH$ are respectively equal to FA, AG, GF ;

$\therefore \triangle s HDK, FAG$ are equal in all respects; (I. 5)

and $\therefore \angle HDK$ is $= \angle FAG$.

PROBLEM F.

Find the shortest path from a given point to a given straight line.



Let A be the given point and BC the given straight line.

From A let fall $AQ \perp$ to BC .

Then shall AQ be less than any other straight line AR drawn from A to BC .

For since $\angle AQR$ is a right angle,

$$\therefore \angle ARQ \text{ is } < \text{ a right angle ; } \quad (I. 13)$$

$$\therefore \angle ARQ \text{ is } < \angle AQR ;$$

$$\therefore AQ \text{ is } < AR. \quad (I. 15)$$

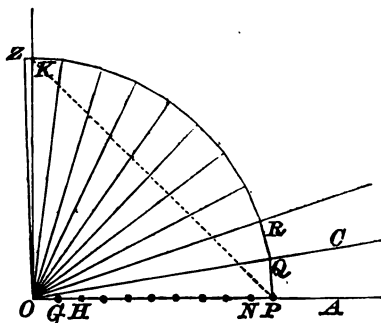
Hence AQ is the shortest path from A to BC .

NOTE. By the distance of a point from a straight line is meant the shortest path from the point to the line.

Hence the distance of a point from a straight line is the perpendicular let fall upon it from the given point.

DEDUCTION G.

If points be taken along one of the arms of an angle farther and farther from the vertex, their distances from the other arm will at length be greater than any given straight line.



Let $\angle AOC$ be the given angle,
then a certain number of these angles $\angle AOC$, $\angle COR$, &c.
will be together $>$ a right $\angle AOK$.

From OA cut off that same number of parts OG , GH , ...
 NP , each equal to the given straight line.

Draw $PQ \perp$ to OC ,
 PQ shall be $> OG$.

Let the $\triangle OPQ$ be turned about OQ into the position
 ORQ , then about OR and so on, till the number of these
 $\triangle$ s is equal to the number of parts in OP ;

thus the $\angle ZOA$ is $>$ the right $\angle KOA$;
also $\angle PQRZ$ is $> OP$, for $\angle PQRK$ is $> PK$, and $\therefore > OP$;
 $\therefore PQ$ is $> OG$.

PARALLEL STRAIGHT LINES.

AXIOM. *If one straight line be drawn in the same plane as another, it cannot first recede from and then approach to the other, neither can it first approach to and then recede from the other on the same side of it.*

INTRODUCTORY LEMMA.

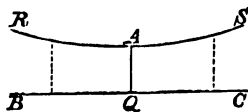
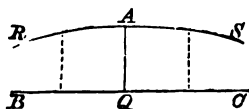
Through a given point without a given straight line one and only one straight line can be drawn in the same plane with the former which shall never meet it.

Also all the points in each of these straight lines are equidistant from the other.

Let A be the given point without the given straight line BC .

From A draw $AQ \perp$ to BC , and through A draw $RAS \perp$ to AQ .

Then the figure may be turned about AQ till QC falls along QB and AS along AR .



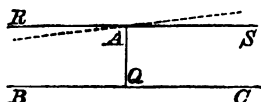
Hence, if there are points in RS on one side of A nearer to BC than A is, so also are there corresponding points on the other side of A nearer to BC than A is; which is impossible.

And if points in RS on one side of A are farther from BC than A is, so also are there corresponding points on the other side of A farther from BC than A is; which is impossible.

$\therefore$ all points in RS are equidistant from BC .

Similarly all points in BC are equidistant from RS .

Again, if any line be drawn through A other than RAS , and points taken along that part of it which is on the



same side of RAS as BC is, farther and farther from A , their distances from RAS will at length be greater than AQ : (I. G.)

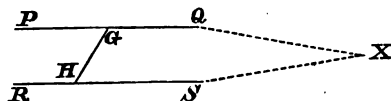
$\therefore$ the straight line thus drawn will at length cut BC .

Hence through A no other straight line besides RAS can be drawn which shall never meet BC .

DEF. Straight lines, lying in the same plane and which will never meet, though produced ever so far both ways, are called **parallel**. [The name being due to the property established in the foregoing Lemma.]

PROPOSITION XX.

If, in a plane, a straight line, meeting two other straight lines, makes the alternate angles equal to one another, these two straight lines shall be parallel.



Let the straight line GH , meeting the two straight lines PQ , RS , make the alternate $\angle$ s PGH , GHS equal to one another.

Then shall PQ be $\parallel$ to RS .

For, if not, PQ , RS when produced will meet either towards QS or towards PR .

Let them, if possible, be produced to meet towards Q , S in the point X .

Then GHX is a Δ ;

and the exterior $\angle PGH$ is $>$ the interior and opposite $\angle GHX$; (I. 13)

but it is also equal to it, which is impossible.

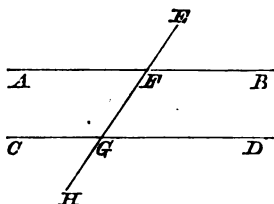
$\therefore PQ$, RS do not meet when produced towards Q , S .

Similarly it may be proved that they will not meet if produced towards P , R .

$\therefore$ they are parallel.

Hence are easily deduced the following Theorems :

If a straight line, falling upon two other straight lines, makes the exterior angle equal to the interior and opposite upon the same side of the line, the two straight lines shall be parallel.



Let the straight line $EFGH$, falling upon the two straight lines AB , CD , make the exterior angle $EFB =$ the interior and opposite $\angle FGD$ upon the same side of EF .

Then shall AB be $\parallel$ to CD .

For the $\angle EFB$ is = the vertical $\angle AFG$; (1. 6)
 $\therefore \angle AFG$ is = the $\angle FGD$, and these are alternate angles;
 $\therefore AB$ is $\parallel$ to CD .

Also

If a straight line falling upon two other straight lines makes the interior angles upon the same side of the line together equal to two right $\angle$ s, these two straight lines shall be parallel.

For let the $\angle$ s BFG , FGD be together equal to right $\angle$ s;

$\therefore$ they are together = $\angle$ s BFG , AFG : (1. 9)

take away the common $\angle BFG$;

$\therefore \angle FGD$ is = $\angle AFG$, and these are alternate angles;

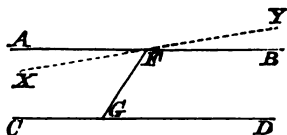
$\therefore AB$ is $\parallel$ to CD .

PROPOSITION XXI.

If a straight line meets two parallel straight lines it makes the alternate angles equal to one another.

Let the straight line FE meet the two $\parallel$ straight lines AB , CD .

Then shall $\angle AFG$ be = the alternate $\angle FGD$.



For if not, if possible, at F in FG draw another line XFY , making $\angle XFG = \angle FGD$. (I. 20)

Then XFY is $\parallel$ to CD .

$\therefore$ through F two straight lines have been drawn $\parallel$ to CD , which is impossible. (Lemma.)

$\therefore \angle AFG$ is = the alternate $\angle FGD$.

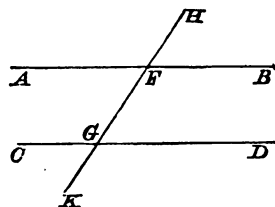
Hence are easily deduced the following Theorems:

If a straight line fall upon two parallel straight lines, it makes the exterior $\angle$ = the interior and opposite upon the same side.

Let HK fall upon the two $\parallel$ straight lines AB , CD ;

then shall the $\angle HFB$ be = $\angle FGD$.

For $\angle AFG$ is = the alternate $\angle FGD$.



And $\angle AFG$ is = the vertically opposite $\angle HFB$; (I. 6)

$\therefore \angle HFB$ is = $\angle FGD$.

If a straight line fall upon two parallel straight lines, the two interior angles upon the same side shall together be equal to two right angles.

That is, the $\angle$ s BFG , FGD shall be together = two right angles.

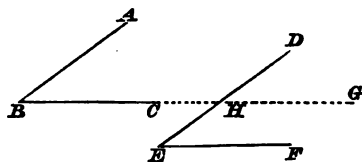
For $\angle AFG$ is = the $\angle FGD$,

and $\therefore \angle$ s AFG , BFG are together = $\angle$ s FGD , BFG .

But $\angle$ s AFG , BFG are together = two right $\angle$ s; (I. 9)

$\therefore \angle$ s FGD , BFG are together = two right $\angle$ s.

COR. *If AB , BC are two straight lines respectively $\parallel$ to DE , EF , then shall $\angle ABC$ be = $\angle DEF$.*

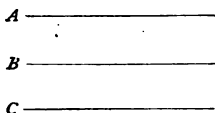


For, let BC , ED , produced, if necessary, intersect in H .

Then $\angle$ s ABC , DEF are each = $\angle DHG$, and are therefore equal to one another.

PROPOSITION XXII.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines A, B be each of them $\parallel$ to C .

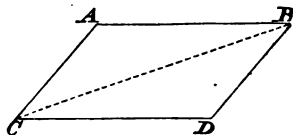
Then shall A, B be $\parallel$ to one another.



For if not they will meet if produced, and there will thus be drawn through one point two straight lines parallel to the same straight line, which is impossible. (Lemma.)

PROPOSITION XXIII.

Straight lines which join the extremities of equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB be = and $\parallel$ to CD .

Join AC, BD .

Then shall AC be = and $\parallel$ to BD .

Join BC .

Then $\because AB$ is $\parallel$ to CD ; $\therefore \angle ABC$ is = $\angle DCB$, (I. 21)

and $\because$ in the Δ s ABC, DCB ,

AB, BC , and the included $\angle ABC$, are respectively
= DC, CB , and $\angle DCB$;

$\therefore$ the Δ s ABC, DCB are equal in all respects. (I. 1)

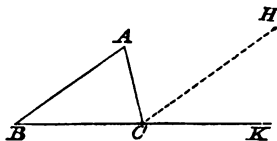
Hence AC is = DB .

Also $\angle ACB$ is = $\angle DBC$,

and $\therefore AC$ is $\parallel$ to BD . (I. 20)

PROPOSITION XXIV.

If one side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles; also the three interior angles of every triangle are together equal to two right angles.



Let ABC be a Δ ,
 and let one of its sides BC be produced to K .
 Then shall $\angle ACK$ be equal to the angles ABC , BAC together;
 also the three interior angles of the ΔABC shall be together equal to two right angles.

Through C draw $CH \parallel$ to AB .

Then $\because CH$ is $\parallel$ to AB , and AC meets them, therefore $\angle ACH$ is = the alternate $\angle BAC$. (I. 21)

Again, $\because CH$ is $\parallel$ to AB , and BK falls upon them;

$\therefore$ the exterior $\angle HCK$ is = the interior and opposite $\angle ABC$; (I. 21)

$\therefore$ the whole $\angle ACK$ is = the two angles ABC , BAC together.

Now to each of these equals add the $\angle ACB$;

$\therefore$ the two angles ACK , ACB are together = the three $\angle$ s of the ΔABC ;

but the two angles ACK , ACB are together = two right $\angle$ s; (I. 9)

$\therefore$ the three $\angle$ s of ΔABC are together = two right $\angle$ s.

PROPOSITION XXV.

If two angles of one triangle are equal to two angles of another, then shall the remaining angle of the one be equal to the remaining angle of the other.

For the three angles of each Δ are together = two right $\angle$ s; (I. 24)

$\therefore$ the three angles of the one are together = the three $\angle$ s of the other ;

but two $\angle$ s of the one are = two $\angle$ s of the other ;

$\therefore$ the remaining $\angle$ of the one is = the remaining $\angle$ of the other.

COR. *If two angles of one triangle are equal to two angles of another, and any side of one is = the corresponding side of the other ; then shall the Δ s be equal in all respects.*

For, by the proposition, the remaining $\angle$ s are = one another.

Hence there are two angles and the side adjacent in the one triangle equal to two angles and the side adjacent in the other ;

$\therefore$ the Δ s are equal in all respects. (I. 3)

QUADRILATERAL AND MULTILATERAL FIGURES.

DEFINITIONS.

Rectilinear figures are those contained by straight lines.

Quadrilateral figures by four straight lines.

Multilateral figures, or polygons, by more than four straight lines.

A parallelogram is a four-sided figure, of which the opposite sides are parallel.

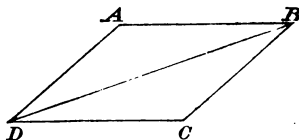
The straight line joining the opposite angles of a quadrilateral figure is called **a diagonal**.

PROPOSITION XXVI.

The opposite sides and angles of a parallelogram are equal, and the diagonal bisects it.

Let $ABCD$ be a $\square$, and BD a diagonal.

Then shall AB be $= CD$,
 $AD = BC$, $\angle BAD = \angle BCD$,
 $\angle ABC = \angle ADC$, and $\triangle$
 $ABD = \triangle CDB$.



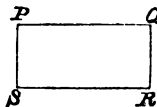
$\therefore AB$ is $\parallel$ to DC , $\therefore \angle ABD$ is $= \angle BDC$, (I. 21)
and $\therefore AD$ is $\parallel$ to BC , $\therefore \angle ADB$ is $= \angle DBC$, (I. 21)
 $\therefore$ the whole $\angle ABC$ is $=$ the whole $\angle ADC$.

Also $\therefore \angle ABD$, $\angle ADB$, and the side BD , are respectively $= \angle CDB$, $\angle CBD$, and side BD ; (I. 3)

$\therefore \triangle$ s ABD , CDB are equal in all respects; $\therefore AB$, AD , and $\angle BAD$ are respectively $= CD$, CB , and $\angle BCD$.

PROPOSITION XXVII.

If one angle of a parallelogram is a right angle, all its angles are right angles.



Let $PQRS$ be a $\square$, having the $\angle PSR$ a right $\angle$.

Then shall the other angles of $PQRS$ be right $\angle$ s.

$\therefore PS$ is $\parallel$ to QR ;

$\therefore \angle s\ PSR, QRS$ are together = two right $\angle$ s : (1. 21)

but $\angle PSR$ is a right $\angle$;

$\therefore$ also $\angle QRS$ is a right $\angle$;

$\therefore$ the $\angle s\ PQR, SPQ$ opposite to these are right $\angle$ s. (1. 26)

DEFINITION.

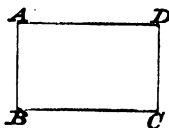
A rectangle is a right-angled parallelogram, and is said to be contained by any two of its adjacent sides.

DEFINITION.

A **square** is a four-sided figure, having all its sides equal to one another, and all its angles right angles.

PROPOSITION XXVIII.

If two adjacent sides of a rectangle are equal, the figure is a square.



Let $ABCD$ be the rectangle contained by AB , AD ,
and let AB be $= AD$.

Then $ABCD$ is a square.

For since $ABCD$ is a $\square$, $\therefore AD$ is $= BC$, (I. 26)

and AB is $= DC$, (I. 26)

also, by hypothesis, AB is $= AD$;

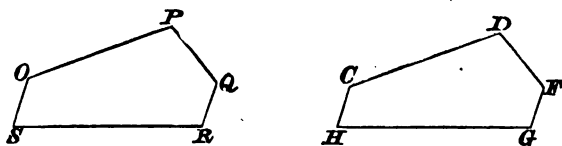
$\therefore ABCD$ is equilateral.

It is also rectangular;

$\therefore ABCD$ is a square.

PROPOSITION XXIX.

If the sides and angles of one rectilinear figure are respectively equal to the corresponding sides and angles of another, the figures are equal in all respects.



Let the sides and angles of the rectilinear figure $OPQRS$ be respectively equal to those of $CDFGH$.

Then shall these figures be equal in all respects.

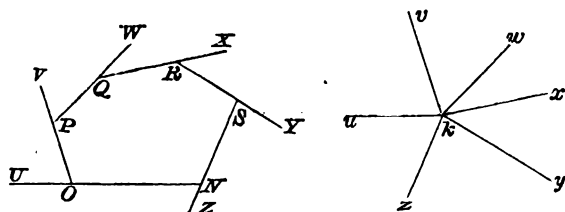
For let the figure $OPQRS$ be applied to $CDFGH$, so that O may fall on C and OP along CD , then P will coincide with D , and $\therefore PQ$ will fall on DE , $\because \angle OPQ$ is $= \angle CDE$, and so on. Thus the figure $OPQRS$ will coincide with $CDFGH$, and will $\therefore$ be $=$ it in all respects.

COR. *If two adjacent sides and the included $\angle$ of one $\square$ are respectively equal to two adjacent sides and the included $\angle$ of another $\square$, then these $\square$ s are equal in all respects; therefore also if two adjacent sides of one rectangle are equal respectively to two adjacent sides of another, the rectangles are equal in all respects.*

N.B. If a, b are two straight lines, then a rectangle having two of its adjacent sides respectively $= a, b$ will be spoken of as the "rectangle contained by (a, b) ," which will be written thus, "rect. (a, b) ."

PROPOSITION XXX.

If the sides of a polygon be produced in order, the exterior angles shall together be equal to four right angles.



Let the sides of the polygon *NOPQRS* be produced in order,

Then shall the exterior angles be together = 4 right $\angle$ s.

Through any point *K* draw *Ku*, *Kv*, *Kw*, &c. $\parallel$ to *OU*, *PV*, *QW*, &c.

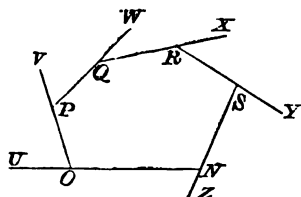
Then the exterior $\angle$ s *UOV*, *VPW*, *WQX*, &c. of the polygon are respectively = the $\angle$ s *uKv*, *vKw*, *wKu*, &c.,

(i. 21)

and are $\therefore$ together = four right $\angle$ s.

(i. 9)

COR. *All the interior angles of a polygon together with four right angles are equal to twice as many right angles as the figure has sides.*



For the $\angle NOP$ of the polygon together with the exterior $\angle UOV$ makes up two right $\angle$ s. (I. 9)

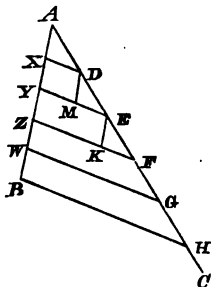
Hence all the interior $\angle$ s of the polygon together with the adjacent exterior $\angle$ s are equal to twice as many right $\angle$ s as the figure has sides ;

but all the exterior $\angle$ s are = 4 right $\angle$ s ;

$\therefore$ all the interior $\angle$ s of the polygon together with four right $\angle$ s are = twice as many right $\angle$ s as the figures has sides.

PROBLEM H.

To divide a straight line into any given number of equal parts.



Let it be required to divide the straight line AB into five equal parts.

Through A draw any other straight line AC :
 from it cut off any part AD ,
 and also DE, EF, FG, GH each equal AD ;
 join HB , and through D, E, F, G draw lines $\parallel$ to HB
 and cutting AB in X, Y, Z, W ;
 then shall AB be divided into five equal parts.

Draw $DM, EK \parallel$ to AB .

Then the $\angle$ s of $\triangle DME$ are = the $\angle$ s of $\triangle AXD$; (I. 31)
 and DE is = AD .

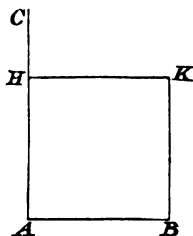
$\therefore DM = AX$ (I. 3); $\therefore XY = AX$. (I. 26)

Similarly YZ , &c. are each = AX ,
 and $\therefore AB$ is divided into five equal parts.

In like manner may a straight line be divided into any given number of equal parts.

PROBLEM K.

Describe a square on a given straight line.



Let AB be the given straight line.
It is required to describe a square on AB .

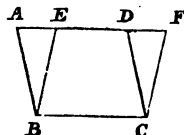
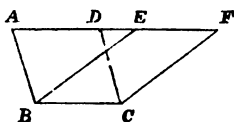
Through A draw $AC \perp$ to AB ,
from AC cut off $AH = AB$.
Through H draw $HK \parallel$ to AB ,
and through B draw $BK \parallel$ to AH .
Then $ABKH$ shall be the square required.

For, by construction, AK is a $\square$.
 $\therefore AB$ is $= HK$, (1. 26)
 and $AH = BK$, (1. 26)
 also AH is $= AB$;
 $\therefore AK$ is equilateral;
 also $\because \angle BAH$ is a right $\angle$;
 $\therefore$ all the $\angle$ s of AK are right $\angle$ s; (1. 27)
 $\therefore$ the $\square AK$ is both equilateral and rectangular,
 and is $\therefore$ a square.

AREAS.

PROPOSITION XXXI.

Parallelograms on the same base and between the same parallels are equal.



Let the $\square$ s $ABCD$, $EBCF$ be on the same base BC , and between the same parallels AF , BC .

Then shall the $\square$ $ABCD$ be = the $\square$ $EBCF$.

$$\therefore AC \text{ is a } \square, \therefore AD = BC; \quad (\text{I. 26})$$

$$\text{and } \therefore EC \text{ is a } \square, \therefore EF = BC; \quad (\text{I. 26})$$

$$\therefore AD = EF;$$

$$\therefore AE = DF.$$

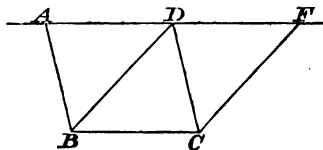
$$\text{Also } AB = DC \text{ and } BE = CF;$$

$$\therefore \triangle ABE = \triangle DCF.$$

Now from the quadrilateral $ABCF$ take the $\triangle ABE$,

and from the same figure take the $\triangle DCF$;

$\therefore$ the remainder, the $\square$ $EBCF$, is = the $\square$ $ABCD$.



If the sides opposite the base terminate in the same point D ,

then each of the $\square$ s is double of the $\triangle DBC$; (I. 26)

$$\therefore \square ABCD \text{ is } = \square DBCF.$$

DEFINITIONS.

The greatest perpendicular which can be let fall from a point in the boundary of a figure on the base or base produced is called the **altitude** of the figure.

The perpendicular let fall on the base or base produced of a Δ from the vertex is called the **altitude** of the Δ .

Perpendiculars let fall on the base or base produced of a $\square$ from points in the opposite side are equal to one another, and any one of them is spoken of as the **altitude** of the $\square$.

If any one of the sides of a rectangle be taken as the base, then either of the adjacent sides is equal to the altitude.

$\therefore$ rectangles, whose bases and altitudes are equal, are themselves equal in all respects. (I. 29)

COROLLARIES TO PROPOSITION XXXI.

A $\square$ is = the rectangle on the same base and between the same parallels.

i.e. a $\square$ is = the rectangle having the same base and altitude; but all rectangles having equal bases and altitudes are equal.

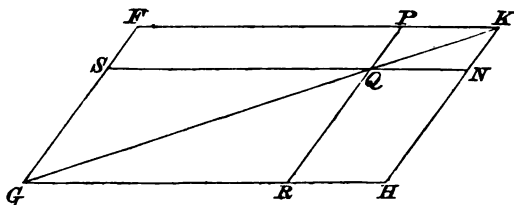
$\therefore$ a $\square$ is = any rectangle of equal base and altitude.

$\therefore$ all parallelograms, having equal bases and altitudes, are equal.

Thus parallelograms upon equal bases and between the same parallels are equal.

PROPOSITION XXXIV.

The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.



Let $FGHK$ be a $\square$, and GK one of its diagonals, and SR, PN $\square$ s about the diagonal.

Then FQ, QH , the remaining $\square$ s, which are called complements, shall be equal to one another.

For $\because$ $\square$ s are bisected by their diagonals, (I. 26)

$\therefore \triangle SGQ$ is $= \triangle RGQ$, and $\triangle PQQ = \triangle NQK$;

$\therefore \triangle s SGQ, PQQ$ are together $= \triangle s RGQ, NQK$;

but the whole $\triangle FGK$ is $=$ the $\triangle HGK$;

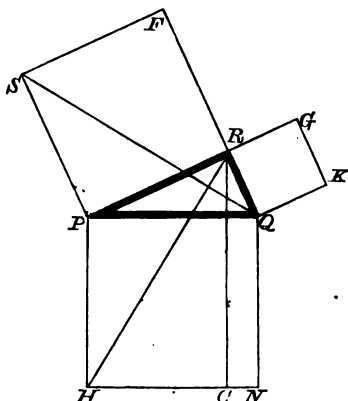
$\therefore$ the remainder FQ is $=$ the remainder QH .

PROPOSITION XXXV.

In any right-angled triangle, the square described on the side opposite the right angle is equal to the squares described on the sides containing the right angle.

Let PN , PF , RK be squares described on the sides of the right-angled $\triangle PQR$.

Then shall PN the square on the side opposite the right $\angle$ be equal to PF , RK .



Through R draw AC
 $\parallel$ to PH ; join RH , SQ .

Now the right $\angle SPR$
 is = the right $\angle QPH$; add to each the $\angle RPQ$;
 $\therefore$ the $\angle SPQ$ is = the $\angle RPH$.

Hence SP , PQ and the $\angle SPQ$ are respectively = RP , PH , and the $\angle RPH$;

$\therefore$ the $\triangle SPQ$ is = the $\triangle RPH$. (I. 1)

Now the $\square PC$ is double of the $\triangle RPH$, because they are on the same base and between the same parallels. (I. 32)

Also $\therefore FRP$ and QRP are right $\angle$ s;

$\therefore FRQ$ is a straight line (I. 10), and it is $\parallel$ to SP .

$\therefore \square SPRF$ is double of the $\triangle SPQ$, (I. 32)

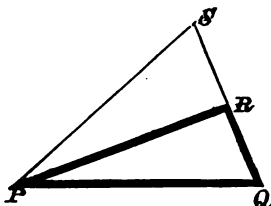
$\therefore \square PC$ is = $SPRF$ the square on PR .

Similarly it may be proved that the $\square QC$ is = the square on RQ ;

$\therefore$ the whole figure $PHNQ$, that is the square on PQ ,
 is = the squares on PR , RQ .

PROPOSITION XXXVI.

If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by those sides is a right angle.



Let the square described on the side PQ of the $\triangle PQR$ be equal to the squares on PR , RQ .

Then shall the $\angle PRQ$ be a right $\angle$.

Through R draw $RS \perp$ to PR and $= RQ$.

Join SP .

Now $\because SR$ is $= QR$; $\therefore$ square on SR is $=$ square on QR .

$\therefore$ the squares on SR , RP are $=$ the squares on QR , RP ,
but the squares on SR , RP are together $=$ the square on SP ;

$\therefore SRP$ is a right $\angle$; (I. 35)

and the squares on QR , RP are $=$ the square on QP ; (hyp.)

$\therefore$ the square on SP is $=$ the square on QP ;

$\therefore SP$ is $= QP$.

Hence, PR , RS , SP are respectively $= PR$, RQ , QP .

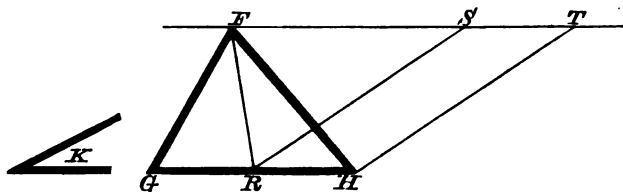
$\therefore$ the $\angle PRS$ is $=$ the $\angle PRQ$, (I. 5)

but $\angle PRS$ is a right $\angle$;

$\therefore \angle PRQ$ is a right $\angle$.

PROBLEM L.

Describe a parallelogram equal to a given triangle, and having an angle equal to a given angle.



Let FGH be the given Δ , and K the given angle.

It is required to describe a $\square = \Delta FGH$ and having an angle $= K$.

Bisect GH in R ; join FR ; through F draw $FST \parallel$ to GH ; at R in RH make $\angle HRS = K$.

Through H draw $HT \parallel$ to RS ;

then $SRHT$ shall be the $\square$ required.

$\therefore GR$ is $= RH$; $\therefore \Delta FGR$ is $= \Delta FRH$; (1. 32)

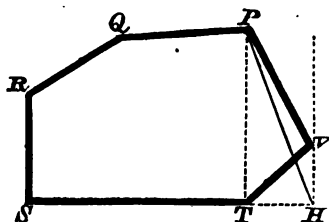
$\therefore \Delta FGH$ is double of the ΔFRH ;

also the $\square SH$ is double of ΔFRH , $\therefore$ they are upon the same base and between the same parallels. (1. 32)

$\therefore \square SH$ is $= \Delta FGH$, and it has an $\angle SRH$ equal to the given angle K .

PROBLEM N.

Describe a triangle equal to a given rectilineal figure.



Let $PQRSTV$ be the given rectilineal figure.

It is required to describe a Δ equal to it.

Join PT : and draw $VH \parallel$ to PT , meeting ST produced in H .

Join PH .

Then ΔPTH is $= \Delta PVT$; (I. 32)

add to each the figure $PQRST$;

$\therefore$ figure $PQRSH$ is $=$ the figure $PQRSTV$.

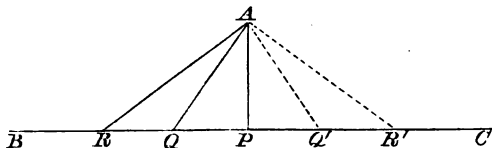
Similarly we may obtain a figure $= RH$ having one fewer sides, and so on; thus we may obtain a Δ equal to the given figure.

COR. Hence a parallelogram may be drawn $=$ a given rectilineal figure and having an angle $=$ a given angle.

SUPPLEMENT TO BOOK I.

THEOREM (a).

Of all straight lines which can be drawn to a given straight line from a given point without it, that which is nearer to the perpendicular is less than the more remote; also to every straight line drawn on one side of the perpendicular there can be drawn one and only one straight line equal to it on the other side.



Let AP be the perpendicular from A on BC ; AQ , AR other straight lines on the same side of AP .

Then $\therefore \angle AQP$ is $<$ a right $\angle$; (I. 13)

$\therefore \angle AQR$ is $>$ a right $\angle$, (I. 9)

and $\angle ARQ$ is $<$ a right $\angle$; (I. 13)

$\therefore \angle ARQ$ is $<$ $\angle AQR$,

$\therefore AQ$ is $<$ AR . (I. 15)

Also if the figure $APQRB$ revolve about AP ,
then PB will fall on PC and Q , R on points Q' , R' in the line PC ,

and $\therefore AQ$, AR on AQ' , AR' ;

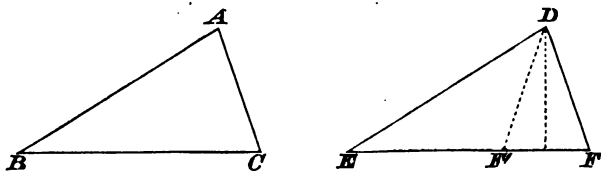
so that to every straight line on one side of AP an equal straight line can be drawn on the other side;

and $\therefore$ no two straight lines on the same side of the perpendicular can be equal;

$\therefore$ there cannot be drawn from it more than two straight lines equal to one another.

THEOREM (b).

If in any $\triangle$ two sides and an $\angle$ opposite to one of them are respectively equal to the corresponding sides and $\angle$ s in another $\triangle$, then shall these $\triangle$ s be equal in certain cases.



Let ABC, DEF be two $\triangle$ s having
 the sides BA, AC and $\angle ABC$ respectively
 $= ED, DF$ and $\angle DEF$;
 then shall $\triangle ABC$ be $= \triangle DEF$ in certain cases.

Let the $\triangle ABC$ be applied to the $\triangle DEF$ so that AB may be on DE and BC along EF .

Then if $\angle DFE$ is a right $\angle$ no other straight line can
 be drawn from $D = DF$; (I. 12)
 $\therefore AC$ must fall on DE .

Hence $\triangle ABC$ is $= \triangle DEF$ in all respects.

But if $\angle DFE$ is not a right $\angle$, then one other straight
 line DF' , and only one, can be drawn $= DF$;
 $\therefore AC$ must fall on DF or DF' .

Hence, if the $\angle$ s opposite the sides AB, DE are both
 acute or both obtuse,
 then will the $\triangle$ s ABC, DEF be equal in all respects.

DEFINITION.

The locus of a point satisfying a certain geometrical condition is the line or lines composed of all possible positions of that point.

PROBLEM (c).

Find the locus of a point equidistant from two given points.

Let A, B be the two given points.

Bisect AB in P , then P is one point in the locus.

Let Q be any other point in the locus.

Join QA, QB, QP .

Then $\because AP, PQ, QA$ are respectively $= BP, PQ, QB$;

$\therefore \angle APQ = \angle BPQ$; (1.5)

$\therefore QP$ is $\perp AB$;

$\therefore Q$ lies on the line bisecting AB at right $\angle$ s.

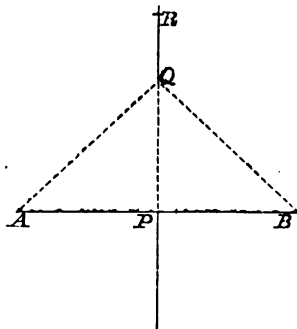
Similarly every other point equidistant from A and B may be shewn to lie on the line bisecting AB at right $\angle$ s; i.e. the locus consists of points which lie on this line.

Moreover, every point in the line bisecting AB at right angles is a point of the locus.

For if any point R be taken in this line, then AP, PR , and the $\angle APR$ are respectively $= BP, PR$, and the $\angle BPR$;

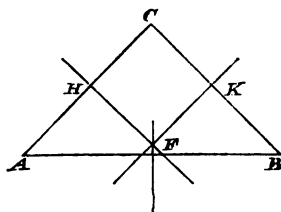
$\therefore R$ is equidistant from A and B and is, $\therefore$ a point of the locus.

Hence the locus of the point equidistant from A and B is the straight line bisecting AB at right $\angle$ s.



DEDUCTION (d).

The lines bisecting the sides of a Δ at right $\angle$ s meet in a point.



Let ABC be a Δ , bisect AC , BC in H , K ,
and draw $HF \perp AC$ and $KF \perp BC$.

Let these perpendiculars meet in F .

Then HF is the locus of points equidistant from A and C ;

$\therefore F$ is equidistant from A and C .

Similarly F is equidistant from B and C ;

$\therefore F$ is equidistant from A and B ;

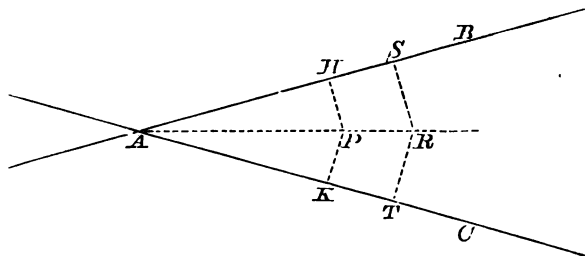
$\therefore F$ lies on the line bisecting AB at right $\angle$ s;

$\therefore$ the line bisecting AB at right $\angle$ s passes through F ;

$\therefore$ the lines bisecting at right $\angle$ s the sides of the ΔABC meet in the point F .

PROBLEM (e).

Find the locus of a point equidistant from two straight lines which cut one another.



Let AB, AC be the two given straight lines,

P a point equidistant from them.

From P let fall PH, PK perpendiculars on AB, AC .

Join AP .

Then $\therefore$ in $\triangle s AHP, AKP$,
 HP, PA , and $\angle AHP$ are respectively $= KP, PA$, and $\angle AKP$,
 and $\angle s HAP, KAP$ are both acute;

$\therefore \angle HAP = \angle KAP$;

$\therefore P$ lies on the line bisecting $\angle BAC$.

Similarly it may be shewn that every other point which is equidistant from AB, AC , lies on the line bisecting $\angle BAC$.

Also every point in the line bisecting the $\angle BAC$ is equidistant from AB and AC .

For if any point R be taken in this line, then letting fall $RS, RT \perp$ on AB, AC , we have

$\angle s RAS, RSA = \angle s RAT, RTA$,

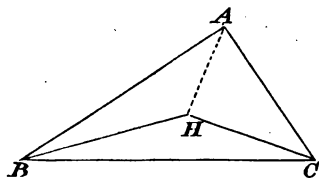
and AR is common to the two $\triangle s ARS, ART$;

$\therefore RS = RT$, or R is equidistant from AB and AC ;

$\therefore$ the locus of the point equidistant from AB, AC , is the line bisecting the angle between them.

DEDUCTION (f).

The bisectors of the $\angle$ s of a $\triangle$ meet in a point.



Let ABC be a $\triangle$,

and bisect the $\angle$ s ABC, ACB by the straight lines BH, CH intersecting in H .

Then BH is the locus of points equidistant from AB, BC ;

$\therefore H$ is equally distant from AB and BC .

Similarly it is equally distant from BC and CA ,

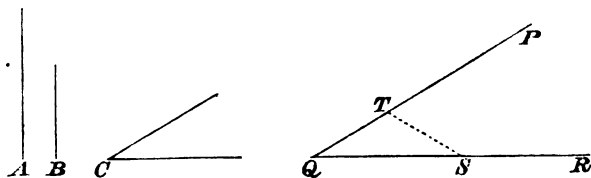
and is $\therefore$ equally distant from AB and AC ,

and $\therefore$ lies on the line bisecting the $\angle BAC$;

$\therefore$ the lines bisecting the $\angle$ s of the $\triangle ABC$ meet in the point H .

PROBLEM (g).

To describe a triangle having two of its sides equal to two given straight lines, and the $\angle$ contained by those sides equal to a given $\angle$.



Let A, B be the given straight lines and C the given angle.

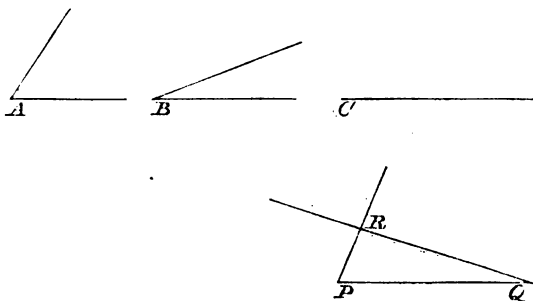
Draw $\angle PQR = \angle C$,

cut off $QS = A$ and $QT = B$; join ST .

Then QST is the triangle required.

PROBLEM (h).

To describe a triangle having two of its angles equal to two given angles, and their common arm equal to a given straight line.



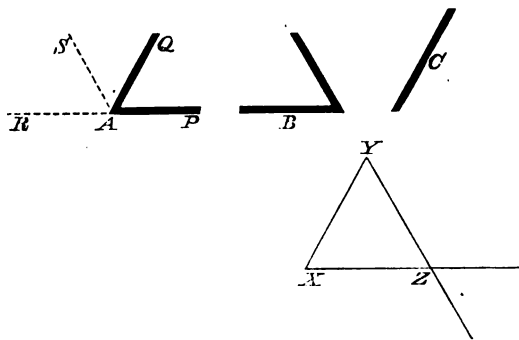
Let A, B be the given angles and C the given straight line. Take $PQ = C$, at P and Q make $\angle$ s $QPR, PQR = A$ and B .

Then PQR is the required Δ .

N.B. This problem is impossible if the two given angles are not together less than two right $\angle$ s.

PROBLEM (k).

To describe a triangle having two of its $\angle$ s equal to two given angles, and the side opposite to one of them equal to a given straight line.



Let PAQ and B be the two given $\angle$ s, and C the given straight line;

produce PA to R and make $\angle RAS = \angle B$.

Hence $\angle QAS$ must be = the remaining $\angle$ of the required Δ .

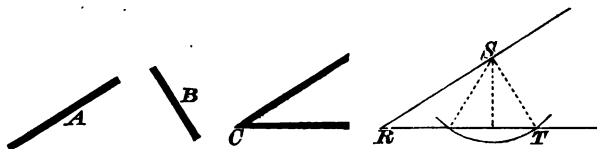
Take $XY = C$, and make $\angle s YXZ, XYZ = \angle s QAP, QAS$.

Then XYZ is the required Δ .

N.B. This problem is impossible if the two given angles are not together less than two right angles.

PROBLEM (I).

To describe a triangle having two of its sides equal to two given straight lines, and the $\angle$ opposite to one of them equal to a given $\angle$.



Let A, B be the given straight lines and C the given $\angle$.

It is required to describe a $\triangle$ having two of its sides equal to A, B and the $\angle$ opposite the latter = C .

Make $\angle SRT = \angle C$, and cut off $RS = A$,
and with centre S and radius = B describe a $\odot$.

If this $\odot$ meet RT in T , join ST .

Then the $\triangle RST$ will be one which satisfies the conditions of the question.

N.B. When C is either an obtuse $\angle$ or a rt. $\angle$ the problem will admit of one solution if B be $> A$; otherwise the problem will be impossible.

When C is an acute angle and B = the perpendicular from S on RT , then only one straight line can be drawn from $S = B$, and there will thus be one solution.

If B be $<$ the perpendicular from S on RT , the problem is impossible.

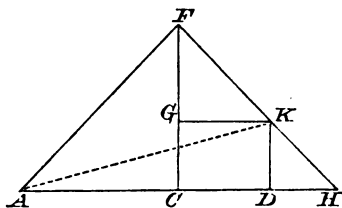
If B be $>$ the perpendicular from S on RT , then two straight lines can be drawn from S to RT , each = B , which will be on the same side of SR if B be $< A$, and thus give two solutions; otherwise, if B be not $< A$, only one solution.

THEOREM (m).

If a straight line be divided into two equal parts and also two unequal parts, the squares on the unequal parts shall together be = twice the squares on half the line and on the line between the points of section.

Let AH be divided into two equal parts in C , and into two unequal parts in D ;

then shall the squares on AD , DH be = twice the squares on AC , CD .



We shall first describe a square = twice the squares on AC , CD .

Draw $CF \perp$ and $= AC$; join AF .

Then ACF is a right-angled isosceles Δ , and square on AF = twice the square on AC . (I. 35)

Join FH ; draw $DK \parallel$ to CF ; $KG \parallel$ to AH ; join AK .

Then it may be easily shewn that FCH , KDH , FGK are right-angled isosceles Δ s, and that $\angle AFH$ is a right $\angle$.

Hence square on FK = twice the square on GK , (I. 35)
= twice the square on CD ; (I. 26)

$\therefore$ squares on AF , FK = twice the squares on AC , CD ;

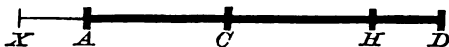
$\therefore$ square on AK = twice the squares on AC , CD .

Also squares on AD , DH = squares on AD , DK ,
= square on AK ; (I. 35)

and $\therefore$ = twice the squares on AC , CD .

COR. Let AH be bisected in C and produced to D : then shall the squares on AD , DH = twice the squares on AC , CD .

This may be established in a similar manner to the above or deduced from it, thus:

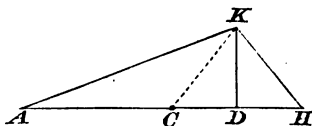


Produce DA and make $AX = HD$. Then $XC = CD$.

$\therefore$ squares on XH , HD = twice the squares on XC , CH ,
i.e. squares on AD , HD = twice the squares on DC , CA .

THEOREM (n).

The squares on the sides of a triangle are double the squares on half the base and on the line joining the vertex and the middle point of the base.



From the vertex K of $\triangle AKH$ draw $KD \perp$ to AH .

Bisect AH in C , and join CK .

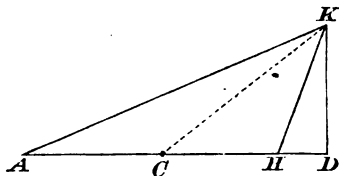
Then shall the squares on $AK, KH =$ twice the squares on AC, CK .

For the squares on $AD, DH =$ twice the squares on AC, CD ;

$\therefore$ the squares on AD, DH and twice the square on $DK =$ twice the square on AC and twice the squares on CD, DK ;

i.e. the squares on $AK, KH =$ twice the squares on AC, CK .

Similarly, if the perpendicular from K fall on AH produced.



Since the squares on $AD, DH =$ twice the squares on AC, CD , it follows that the squares on $AK, KH =$ twice the squares on AC, CK .

Euclid solves the following two problems, the necessity for which has been removed in this treatise by postulate 3.

The corresponding postulate of Euclid is "Let it be granted that a circle may be described from any centre at any distance from that centre," which does not allow the transfer of distances by means of compasses.

PROBLEM (*p*).

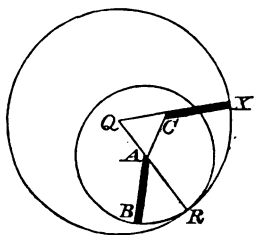
From a given point to draw a straight line equal to a given straight line.

Let C be the given point,
and AB the given straight
line.

Join AC ,

and upon AC describe
an equilateral Δ ;

with centre A and radius
 AB describe a $\odot$ cutting QA
produced in R .



With centre Q and radius QR describe a $\odot$ cutting QC
produced in X .

Then CX shall be the line required.

For $QX = QR$; $\because Q$ is centre of circle RX ;

$QA = QC$; $\because QAC$ is an equilateral Δ ;

$\therefore CX = AR$; but $AR = AB$;

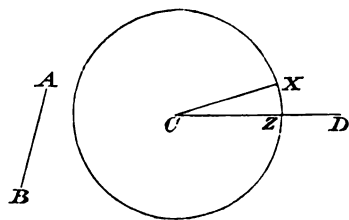
$\therefore CX = AB$;

$\therefore$ from C a straight line CX has been drawn $= AB$.

Q. E. F.

PROBLEM (q).

From the greater of two straight lines to cut off a part equal to the less.



Let AB CD , be the two given straight lines of which CD is the greater.

It is required to cut off from CD a part $= AB$.

From C draw a straight line $CX = AB$.

With centre C and radius CX describe a circle cutting CD in Z .

Then CZ is the part required.

For $CZ = CX$, and $CX = AB$;

$\therefore CZ = AB$.

$\therefore$ from CD a part CZ has been cut off $= AB$.

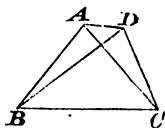
Q. E. F.

THEOREM (r).

Upon the same base and upon the same side of it there cannot be two triangles which have their conterminous sides equal.

For if it be possible upon the same base BC and on the same side of it let there be two Δ s ABC, DBC which have their conterminous sides equal, viz. $AB = DB$ and $AC = DC$.

1st. Let the vertex of each Δ fall without the other Δ .



Join AD .

Then $\because AB = DB$;

$\therefore \angle BAD = \angle BDA$;

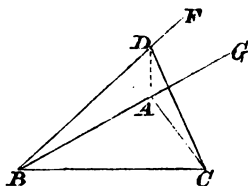
but $\angle BAD$ is $> \angle CAD$; $\therefore \angle BDA$ is $> \angle CAD$;

but CDA is $> \angle BDA$;

much more then is $\angle CDA > \angle CAD$;

but since $AC = DC$, it follows that $\angle CDA = \angle CAD$,
which is impossible.

2nd. Let the vertex of one of the Δ s as A fall within the other ΔDBC .



Join AD ; and produce BD , BA to F , G .

Then $\because AB = DB$;

$\therefore \angle FDA = \angle GAD$;

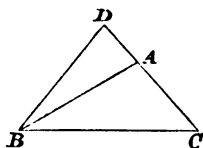
but $\angle FDA > \angle CDA$; $\therefore \angle GAD > \angle CDA$;

but $\angle CAD > \angle GAD$;

much more then is $\angle CAD > \angle CDA$;

but since $AC = DC$; $\therefore \angle CAD$ is $= \angle CDA$, which is impossible.

3rd. Let the vertex A of one of the Δ s fall upon one of the sides of the other.



Then it is evident that AC is $< DC$,

but by hypothesis it is $=$ it, which is impossible.

Wherefore upon the same base, &c.

N.B. By the aid of this Theorem Prop. (i. v.) may easily be established.

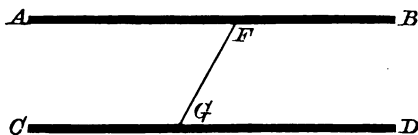
Euclid's Twelfth Axiom, on which his Theory of Parallels depends, is "If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these two straight lines being continually produced, shall at length meet on that side on which are the angles which are less than two right angles."

Hence Propositions XXI. and XXII. are established as follows :

PROPOSITION XXI.

If a straight line meets two parallel straight lines it makes the alternate angles equal to one another.

Let the straight line FG meet the two $\parallel$ straight lines AB , CD , then shall $\angle AFG =$ the alternate $\angle FGD$.



For, if not, one of them must be the greater.

Let $\angle AFG$ be greater than $\angle FGD$;

$\therefore \angle s$ AFG , GFB are together $>$ $\angle s$ FGD , GFB ,

but $\angle s$ AFG , GFB are together = two right $\angle s$;

$\therefore \angle s$ FGD , GFB are together $<$ two right $\angle s$;

$\therefore AB$, CD will meet towards B , D ; but they do not.

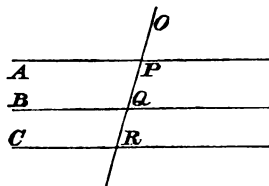
Hence $\angle AFG = \angle FGD$.

PROPOSITION XXII.

Straight lines which are parallel to the same straight line are parallel to one another.

Let the straight lines A, B be each of them $\parallel$ to C ,
then shall A be $\parallel$ to B .

Draw $OPQR$ cutting A, B, C in P, Q, R .



Then the $\angle$ s OPA, OQB are each $= \angle ORC$,
and $\therefore$ = one another.

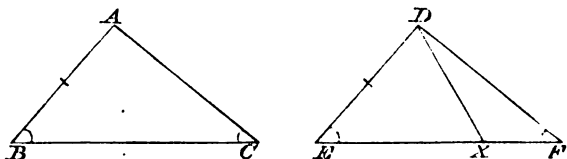
$\therefore A$ is $\parallel$ to B .

PROPOSITION XXV. COR.

If two angles of one triangle are equal to two angles of another, and any side of one is equal to the corresponding side of the other, then shall the triangles be equal in all respects.

The case in which the given sides in each are adjacent to the equal angles is established in Prop. III.

Euclid proves the case in which the given equal sides are opposite to equal angles in each thus :



Let ABC, DEF be two Δ s having $\angle ABC = \angle DEF$,
 $\angle ACB = \angle DFE$, and side $AB =$ side DE .

Then BC and EF are equal to one another,
 for, if not, one of them must be the greater,
 let this be EF , and from EF cut off $EX = BC$; join DX .
 Then $\because AB = DE, BC = EX$, and $\angle ABC = \angle DEX$;

$$\therefore \angle ACB = \angle DXE,$$

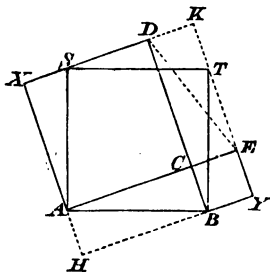
but $\angle ACB = \angle DFE$; $\therefore \angle DXE = \angle DFE$;
 the exterior $\angle$ of a Δ = the interior and opposite angle ;
 which is impossible ;

$$\therefore BC \text{ is } = EF.$$

Also $AB = DE$ and $\angle ABC = \angle DEF$;

$\therefore \Delta ABC$ is $= \Delta DEF$ in all respects.

The celebrated proposition of Pythagoras (I. 35) may be established as follows.



Let ABC be a right-angled Δ having the right $\angle ACB$.

On AC , CB describe squares XC , YC .

Then ACE , BCD are straight lines equal and $\perp$ to one another.

Complete the square XY .

Draw AS , $BT \perp$ to AB , and join ST .

Then Δ s AXS , TYB are each $= \Delta ACB$ in all respects;

$\therefore AS$ is $=$ and $\parallel$ to BT ,

Hence BS is a $\square$, and $\therefore$ the square on AB .

Now, square on AB + four Δ s each equal to ΔABC
 $=$ square XY ;

also squares on AC , CB + four Δ s each equal to ΔABC
 $=$ square XY ;

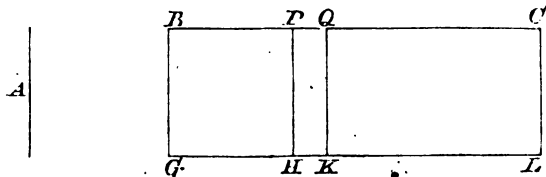
$\therefore$ square on $AB =$ squares on AC , CB .

BOOK II.

RECTANGLES.

PROPOSITION I.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.



Let A and BC be two straight lines,
and let BC be divided into any parts BP , PQ , QC .

Then shall the rectangle A , BC be = rectangle A , BP
+ rectangle A , PQ + rectangle A , QC .

From B draw $BG \perp$ to BC and = A .

From G draw $GL \parallel$ to BC ,
and from P , Q , C draw PH , QK , $CL \parallel$ to BG .

Then BL, BH, PK, QL are $\square$ s ;

also $\angle$ s QPH, CQK are each = $\angle CBG$ which is a right $\angle$; (I. 21)

$\therefore BL, BH, PK, QL$ are rectangles. (I. 27)

Now the figure BL is = the figures BH, PK, QL together ; that is, rectangle BG, BC is = rectangle BG, BP

+ rectangle PH, PQ + rectangle QK, QC .

$\therefore$ rectangle A, BC is = rectangle A, BP

+ rectangle A, PQ + rectangle A, QC .

COR. 1. If A be = BC ,

then the rectangle (A, BC) is = the square on BC .

$\therefore$ square on BC is = rectangle BC, BP + rectangle BC, PQ + rectangle BC, QC .

Hence, *if a straight line be divided into any number of parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and the several parts.*

COR. 2. If A be = one of the parts of the divided line, as BP ,

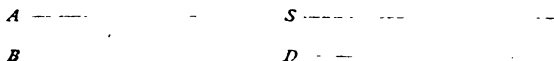
then the rectangle A, BP is = the square on BP .

$\therefore$ rectangle BP, BC = square on BP + rectangle BP, PQ + rectangle BP, QC .

Hence, *if a straight line be divided into any number of parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangles contained by that part and the other parts.*

PROPOSITION II.

If A and B be two straight lines and S their sum, the square on S is $>$ the squares on A and B by twice the rectangle A, B . Also, if A and B be unequal and D be their difference, the rectangle S, D is $=$ the difference between the squares on A, B ; and the square on D is $<$ the squares on A and B by twice the rectangle A, B .



Since S is $= A$ together with B ;

$\therefore$ rectangle $S, A =$ square on A + rectangle A, B , (II. 1)

rectangle $S, B =$ rectangle A, B + square on B , (II. 1)

and square on $S =$ rectangle S, A + rectangle S, B , (II. 1)

$\therefore$ square on $S =$ squares on A, B + twice rectangle A, B .

Again, since $D + B = A$;

$\therefore$ rectangle S, D + rectangle $S, B =$ rectangle S, A , (II. 1)

$\therefore$ rectangle S, D + rectangle A, B + square on B
 $=$ square on A + rectangle A, B ;

$\therefore$ rectangle S, D + square on $B =$ square on A .

Also since $D + B = A$;

$\therefore$ square on $D +$ rectangle $D, B =$ rectangle D, A ; (II. 1)

But rectangle $D, A +$ rectangle $A, B =$ square on A , (II. 1)

and rectangle $D, B +$ square on $B =$ rectangle A, B , (II. 1)

$\therefore$ square on $D +$ rectangle $D, B +$ rectangle A, B

$=$ rectangle $D, A +$ rectangle A, B ,

$\therefore =$ square on A ;

$\therefore$ sq. on $D +$ rect. $D, B +$ sq. on $B +$ rect. $A, B =$ sqq. on A, B ;

$\therefore$ square on $D +$ twice rectangle $A, B =$ squares on A, B .

COR. It follows from the proposition that
squares on $S, D +$ twice rectangle A, B

$=$ twice squares on $A, B +$ twice rectangle A, B ;

$\therefore$ squares on $S, D =$ twice squares on A, B .

Also that

square on $S =$ squares on $A, B +$ twice rectangle A, B ,

and $\therefore =$ square on $D +$ four times rectangle A, B .

N.B. By the method here adopted all propositions which are the Geometrical interpretations of Algebraical identities of the second degree may be established.

DEFINITIONS.

An **obtuse angle** is one greater than a right angle.

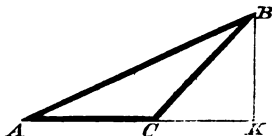
An **acute angle** is one less than a right angle.

An **obtuse-angled triangle** is one having an obtuse angle.

An **acute-angled triangle** is one having three acute angles.

PROPOSITION III.

In obtuse-angled triangles, the square on the side opposite the obtuse angle is greater than the squares on the sides containing it by twice the rectangle contained by either of those sides and the part of it intercepted between the perpendicular let fall upon it from the opposite angle and the obtuse angle.



Let ABC be an obtuse-angled $\triangle$, having the obtuse $\angle ACB$,

and from B let fall $BK \perp$ to AC produced.

Then shall the square on AB be $>$ the squares on AC , CB by twice the rectangle AC , CK .

For AK is = the sum of AC and CK ;

$\therefore$ square on AK is = squares on AC , CK

+ twice rectangle AC , CK ; (II. 2)

$\therefore$ squares on AK , KB are = squares on AC , CK , KB

+ twice rectangle AC , CK ;

but squares on AK , KB are = square on AB ,

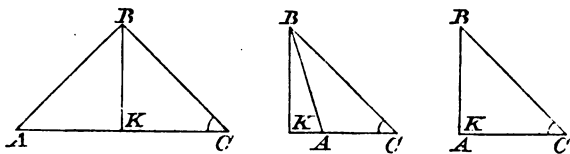
and squares on CK , KB are = square on CB ;

$\therefore$ square on AB is = squares on AC , CB

+ twice rectangle AC , CK .

PROPOSITION IV.

In any triangle the square on the side opposite to any one of the acute angles is less than the squares on the sides containing it by twice the rectangle contained by either of those sides and the part of it intercepted between the perpendicular let fall upon it from the opposite angle and the acute angle.



Let ABC be a $\triangle$ having the acute $\angle ACB$,
and from B let fall $BK \perp CA$ or CA produced.

Then shall the square on AB be $<$ the squares on AC , CB by twice the rectangle AC , CK .

For should the $\perp$ fall within or without the $\triangle ABC$, in either case AK is = the difference between AC and CK .

$\therefore$ square on AK + twice rectangle AC , CK is = squares on AC , CK . (II. 2)

$\therefore$ squares on AK , KB + twice rectangle AC , CK = squares on AC , CK , KB .

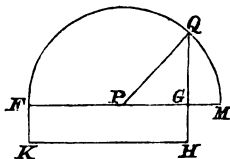
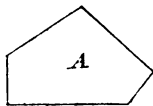
$\therefore$ square on AB + twice rectangle AC , CK = squares on AC , CB .

Should the $\angle BAC$ be a right $\angle$, and $\therefore BK$ coincide with BA ,

then square on AB + twice square on AC = squares on AC , CB . (I. 35)

PROBLEM A.

Describe a square equal to a given rectilineal figure.



Let A be the given rectilineal figure.

First describe the rectangle $FH = A$;

then if the sides are equal FH is a square,
and what was required is done.

But, if not, produce FG and cut off $GM = GH$.

Bisect FM in P ,

and with centre P and radius PM describe $\odot FQM$.

Produce HG to meet the $\odot$ in Q .

The square on GQ shall be $= A$.

Join PQ .

Then $\therefore FG$ is = the sum and GM = the difference of
 PM and PG ;

$\therefore$ rectangle FG , GM is = the difference of the squares
on PM and PG , (11. 2)

and is $\therefore$ = the difference of the squares on PQ and PG ,

and is $\therefore$ = the square on GQ , (1. 35)

but FH is = rectangle FG , GM , for $GH = GM$,

and FH was made $= A$;

$\therefore$ square on $GQ = A$.

PROBLEM B.

To divide a straight line into two parts so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

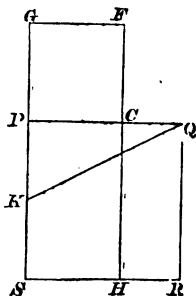
Let PQ be the given straight line.

On PQ describe the square PR .

Bisect PS in K and join QK .

Produce SP and cut off $KG = KQ$.

On PG describe a square $GPCF$,
one side of which PC will fall on PQ .
Then shall PQ be divided in C , so that
rectangle PQ , QC = square on PC .



Produce FC to H .

Now $\therefore SG$ is = the sum and GP = the difference of
 KG and KP ,

$\therefore$ rectangle SG , GP is = the difference of the squares
on KG and KP , (II. 2)

and is $\therefore$ = the difference of the squares on KQ and KP ,
and is $\therefore$ = the square on PQ ; (I. 35)

but figure GH = rectangle SG , GP ; for $GF = GP$,

and figure PR is the square on PQ ;

$\therefore$ figure GH = figure PR .

Take away the common part, the figure PH ,

$\therefore$ the remainder GC = the remainder CR ,

but GC is the square on PC and CR is = rectangle PQ , QC ,
for QR is = PQ ;

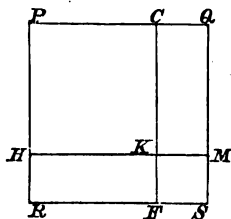
$\therefore$ rectangle PQ , QC = square on PC .

SUPPLEMENT TO BOOK II:

The Theorems in Proposition II. may be established independently as follows :—

THEOREM (a).

If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the parts together with twice the rectangle contained by the parts.



Let the straight line PQ be divided into any two parts in C , then shall square on PQ be = the squares on PC , CQ + twice the rectangle PC , CQ .

Draw $PR \perp$ to PQ and = it;

cut off $PH = PC$ so that $HR = CQ$;

draw HM , $RS \parallel$ to PQ ; and CF , $QS \parallel$ to PR .

Then $PRSQ$ is a square, (1. 28)

and it is = the figures PK , KS , CM , HF .

Now PK is the square on PC , (1. 28)

KS is = square on CQ , for KM is = CQ ; (1. 28)

CM is = rectangle PC , CQ , for CK is = PC ,

and HF is = rectangle PC , CQ , for HK is = PC , and HR is = CQ ;

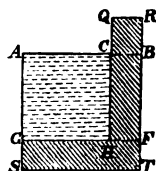
$\therefore$ square on PQ is = squares on PC , CQ + twice rectangle PC , CQ .

THEOREM (b).

If a straight line be divided into any two parts, the square on one part is less than the squares on the whole and on the other part by twice the rectangle contained by the whole and that other part.

Let the straight line AB be divided into any two parts in C .

Then shall the square on AC be $<$ the squares on AB and BC by twice the rectangle AB, BC .



Draw $AGS \perp$ to AB , and $QCH, RBT \parallel$ to AGS ;

cut off $AG = AC, AS = AB$, and $BR = BC$;

draw GHF, ST and $RQ \parallel$ to AB .

Then AH, AT and BQ are squares,
and we have to shew that
 AH is $<$ AT and BQ together by twice rectangle AB, BC ,
i.e. that the figure GTQ is = twice rectangle AB, BC .

Now figure $GT =$ rectangle AB, BC ,
for $GF = AB$ and $GS = BC$; (I. 26)

also figure $QF =$ rectangle AB, BC ,

for $RF = AB$ and $QR = BC$;

$\therefore$ figure $GTQ =$ twice rectangle AB, BC ;

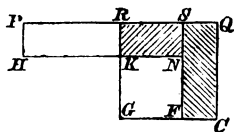
$\therefore$ square on AC is $<$ squares on AB, BC by twice rectangle AB, BC .

THEOREM (c).

The difference between the squares on any two straight lines is equal to the rectangle contained by two straight lines, one of which is equal to the sum, and the other equal to the difference of the given lines.

Let RQ , RS be equal to the given straight lines.

Then SQ is = the difference between them.



Produce QR and make $RP = QR$,
then PS is = the sum of the given lines.

The difference between the squares on RQ , RS shall be = rectangle (PS, SQ) .

On RQ describe a square, draw $PH \perp$ to PQ and = SQ ;
draw $SF \parallel$ to RG and $HKN \parallel$ to PQ .

Then KF is = square on RS , for KN is = RS ;
 $\therefore$ difference between the squares on RQ , RS is = figure KQF .

We have now to shew that figure KQF = rectangle (PS, SQ) .

The rectangle QF = rectangle PK , for QS , QC are respectively = PH , PR ;

$\therefore$ figure KQF = figure PN , which is = rectangle PS, SQ , for $PH = SQ$;

$\therefore$ the difference between the squares on RQ , RS = rectangle (PS, SQ) .

COR. $\overline{P} \quad \overline{R} \quad \overline{S} \quad \overline{Q} \quad \overline{P} \quad \overline{R} \quad \overline{Q} \quad \overline{S}$

If PQ be bisected in R , and any point S be taken in PQ or PQ produced,

then rectangle PS, SQ = difference between the squares on RQ , RS .

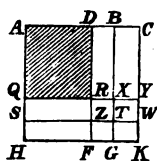
THEOREM (d).

The square on the sum of two straight lines is greater than the square on their difference by four times the rectangle contained by them.

Let the two straight lines AB , BC be placed in the same straight line :

cut off $BD = BC$.

Then shall the square on AC be greater than the square on AD by four times the rectangle (AB, BC) .



On AC , AD describe squares, produce QR , DR :

cut off $QS = BC$, draw $SW \parallel$ to AC and $BG \parallel$ to AH .

The square on AC is greater than the square on AD by figure DKQ .

Hence we have to prove that figure DKQ = four times the rectangle (AB, BC) .

Now the rectangles SF , SR , DX , CX are equal to one another,

also the squares FT , TK , RT , TY are equal to one another;

$\therefore$ figure DKQ = four times the figures SF , FT together;

i.e. four times rectangle (HG, HS) ,

and $\therefore$ = four times rectangle (AB, BC) .

$\therefore$ square on AC is greater than square on AD by four times rectangle (AB, BC) .

BOOK III. THE CIRCLE.

CHORDS.

DEFINITION.

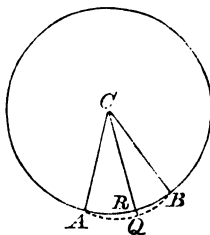
The straight line joining any two points in the circumference of a circle is called a **chord** of the circle.

PROPOSITION I.

The straight line joining any two points in the circumference of a circle lies entirely within the circle.

Let A, B be any two points in the $\odot$ ce of the $\odot ARB$, whose centre is C .

Then shall the straight line joining A, B fall within the $\odot$.



For, if not, if possible, let it fall without the $\odot$ as AQB .

Join C with A and B ;

and draw a straight line CRQ cutting the circle in R ,
and the straight line AB in Q .

Then $\because CA = CB$; $\therefore \angle CAB = \angle CBA$; (I. 2)

but $\angle CQB$ is $> \angle CAB$; (I. 13)

$\therefore \angle CQB$ is $> \angle CBA$;

$\therefore CB$ is $> CQ$; (I. 15)

but $CB = CR$; $\therefore CR$ is $> CQ$, which is impossible.

$\therefore$ the straight line joining A, B does not fall without the $\odot$.

Similarly it may be shewn that it does not fall on the $\odot$ ce;

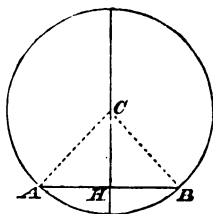
$\therefore$ it falls within the $\odot$.

COR. *A straight line cannot cut a circle in more than two points.*

For if it cut the $\odot$ in more than two points, then the part of the straight line between the extreme points of section would not be entirely within the circle.

PROPOSITION II.

The straight line drawn through the middle point of a chord of a circle and the centre is perpendicular to the chord.



Let AB be a chord of a $\odot$ of which C is the centre.

Then the straight line CH through C and the middle point H of AB shall be $\perp$ to AB .

Join CA, CB .

Then $\therefore AH, HC, CA$ are respectively $= BH, HC, CB$;

$\therefore \angle AHC = \angle BHC$; (I. 5)

$\therefore CH$ is $\perp AB$.

Conversely, *The straight line drawn through the centre of a circle $\perp$ to a chord bisects the chord.*

For through the centre only one straight line can be drawn $\perp$ to the chord; and that through the middle point of the chord is $\perp$ to it.

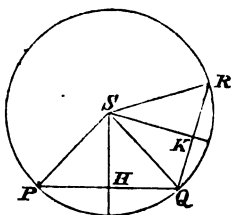
Also, *The straight line drawn through the middle point of a chord of a circle $\perp$ to the chord passes through the centre.*

For through the middle point of the chord only one straight line can be drawn $\perp$ to it; and that through the centre is $\perp$ to it.

COR. *The middle point of a chord bisecting another at right angles is the centre.*

PROPOSITION III.

If from a point within a circle there be drawn to the circumference more than two equal straight lines, that point is the centre of the circle.



Let S be a point within the $\odot PQR$,
from which are drawn to the circumference the equal
straight lines SP, SQ, SR .

S shall be the centre of the circle.

Join PQ, QR ; bisect PQ in H and QR in K .

Draw a straight line through H, S ,

also a straight line through K, S .

Then $\because PH, HS, SP$ are respectively QH, HS, SQ ;

$$\therefore \angle SHP = \angle SHQ; \quad (1. 5)$$

$$\therefore SH \text{ is } \perp \text{ to } PQ.$$

$\therefore$ the centre of the $\odot$ is in the straight line HS . (III. 2)

Similarly it is in the straight line KS ;

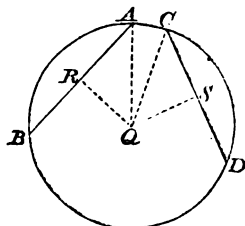
$\therefore S$ is the centre of the $\odot PQR$.

DEFINITION.

Straight lines are said to be **equally distant** from a given **point** when the perpendiculars let fall upon them from the given point are equal; also one straight line is said to be farther than another from a given point when the perpendicular from the given point on the former is greater than that on the latter.

PROPOSITION IV.

Equal chords of a circle are equally distant from the centre.



Let AB , CD be equal chords of the $\odot ABC$.

Then shall AB , CD be equally distant from the centre Q .

Draw $QR \perp$ to AB and $\therefore$ bisecting AB , (III. 2)

and $QS \perp$ to CD and $\therefore$ bisecting CD . (III. 2)

Join AQ , CQ .

Then $\because AB = CD$; $\therefore AR = CS$;

also $\because AQ = CQ$; $\therefore$ square on $AQ =$ square on CQ ;

$\therefore$ squares on AR , $RQ =$ squares on CS , SQ , (I. 35)

but square on $AR =$ square on CS ;

$\therefore$ square on $RQ =$ square on SQ ;

$\therefore RQ = SQ$;

$\therefore AB$, CD are equally distant from the centre of $\odot ABC$.

Conversely:

Chords in a circle which are equally distant from the centre are equal to one another.

Let the chords AB , CD of the $\odot ABC$ be equally distant from the centre Q ;

that is, let the perpendiculars QR , QS drawn to them from Q be equal.

Then shall AB , CD be equal to one another.

Join QA , QC .

Then $\because QR$, QS are $\perp$ to AB , CD respectively;

$\therefore AB$, CD are bisected in R , S . (III. 2)

Also $\because QA = QC$, $\therefore$ square on QA = square on QC ;

$\therefore$ squares on QR , RA are together = squares on QS , SC , (I. 35)

but square on QR = square on QS ; $\because QR = QS$;

$\therefore$ square on RA = square on SC ;

$\therefore RA = SC$;

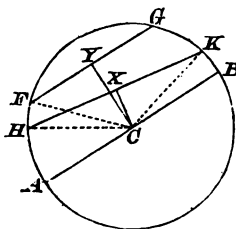
$\therefore AB = CD$.

DEFINITION.

A chord of a circle which passes through the centre is called a **diameter**.

PROPOSITION V.

The diameter is the greatest chord of a circle, and of all others that nearer to the centre is greater than the more remote.



Let ACB be a diameter of the $\odot AGB$,
 HK, FG chords of which HK is nearer to the centre C
 than FG .

Then shall AB be $> HK$ and $HK > FG$.

Draw $CX \perp$ to HK , and $\therefore$ bisecting HK . (III. 2)

Also $CY \perp$ to FG , and $\therefore$ bisecting FG .

Join CH, CK, CF .

Then HC, CK are together $> HK$; (I. 16)

$\therefore AC, CB$ are together $> HK$;

i.e. ACB is $> HK$.

Again, $\therefore CH = CF$; $\therefore$ square on CH = square on CF ;
 $\therefore$ squares on CX, XH are together = squares on CY, YF ; (I. 35)
 but CX is $< CY$, and $\therefore$ square on CX is $<$ square on CY ;

$\therefore$ square on XH is $>$ square on YF ,

and $\therefore XH$ is $> YF$;

$\therefore HK$ is $> FG$.

Conversely:

Of any number of chords in a circle the greater is nearer to the centre than the less.

In the circle AGB let the chord HK be $> FG$.

Then shall HK be nearer to the centre C than FG .

The same construction being made,

$\therefore CH = CF$, $\therefore$ square on CH = square on CF ;

$\therefore$ squares on CX, XH are together = squares on CY, YF ; (I. 35)

but since HK is $> FG$; $\therefore XH$ is $> YF$;

$\therefore$ square on XH is $>$ square on YF ;

$\therefore$ square on CX is $<$ square on CY ;

$\therefore CX$ is $< CY$;

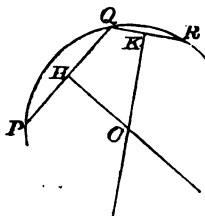
$\therefore HK$ is nearer to the centre C than FG .

DEFINITION.

An **arc** of a circle is a part of the circumference.

PROBLEM A.

An arc of a circle being given, to find the centre of the circle of which it is an arc.



Take any three points P, Q, R in the given arc.

Join PQ, QR :

bisect PQ in H and QR in K .

Through H, K draw straight lines $\perp$ to PQ, QR respectively.

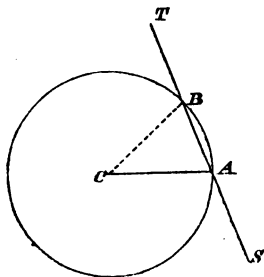
Then the centre of the $\odot$ must lie in each of these straight lines; (III. 2)

and must therefore be their point of intersection.

TANGENTS.

PROPOSITION VI.

The straight line drawn through any point in the circumference of a circle perpendicular to the radius at that point falls without the circle: also through that point no other straight line can be drawn which does not cut the circle.



Let A be any point in the circumference of the $\odot AB$.

Join A with the centre C ;

and through A draw $ST \perp$ to CA .

Then ST shall fall without the $\odot$.

For, if possible, let ST cut the circle in B .

Join BC .

Then $\because CB = CA$;

$\therefore \angle CBA = \angle CAB$; (I. 2)

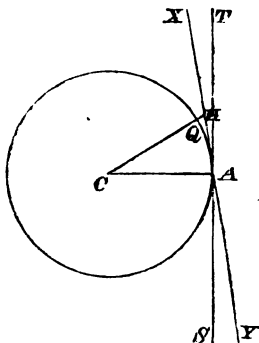
but $\angle CAB$ is a right $\angle$; $\therefore \angle CBA$ is a right $\angle$;

$\therefore$ two of the $\angle$ s of a $\triangle$ are right $\angle$ s, which is impossible; (I. 13)

$\therefore SAT$ does not cut the $\odot$.

Similarly it may be shewn that it has no other point besides A in common with the circumference.

Moreover, besides SAT no other straight line can be drawn through A so as not to cut the circle.



For, if possible, let YAX be so drawn :
let fall $CH \perp$ on YAX cutting the $\odot$ in Q .

Then $\therefore \angle CHA$ is a right $\angle$;

$\therefore CA$ is $> CH$;

but $CA = CQ$;

$\therefore CQ$ is $> CH$, which is impossible.

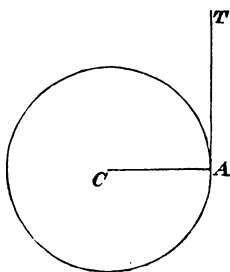
$\therefore$ no straight line besides SAT can be drawn through A so as not to cut the $\odot$.

DEFINITION.

A **straight line** which meets a circle and which, if produced, will not cut it, is said to **touch the circle**.

Conversely:

If a straight line touch a circle the radius through the point of contact is perpendicular to the tangent.



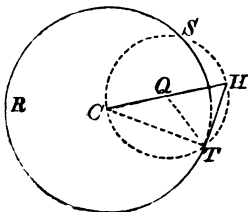
For through the point of contact only one straight line can be drawn touching the circle, and that straight line is $\perp$ to the radius.

Also, If a straight line touch a circle the perpendicular to the tangent drawn from the point of contact shall pass through the centre.

For from the point of contact only one straight line can be drawn $\perp$ to the tangent, and that through the centre is $\perp$ to it.

PROBLEM B.

Draw a tangent to a circle from a given point,



If the given point is on the circumference,
then the straight line drawn $\perp$ to the radius through
that point will be a tangent.

If the given point is without the circle,
join H the given point with C the centre of the given
circle:

bisect CH in Q .

With Q as centre and radius QC or QH describe a
circle cutting the given circle in T .

Join HT ;

then HT shall touch the given circle.

$$\therefore CQ = QT, \therefore \angle QCT = \angle QTC; \quad (I. 2)$$

$$\text{so also } \angle QHT = \angle QTH;$$

$$\therefore \angle s \angle QCT, \angle QHT \text{ are together} = \text{the whole } \angle CTH;$$

$$\text{which is } \therefore \text{ a right } \angle. \quad (I. 24)$$

$$\therefore HT \text{ touches the circle } RST. \quad (III. 6)$$

PROBLEM C.

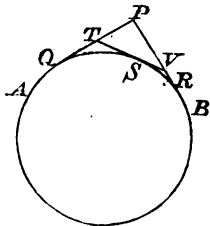
Between two points in the circumference of a circle find the shortest path which does not cut the circle.

Let A, B be two points in the circumference of a $\odot$.

Then the arc AB which is the smaller of the two segments into which A, B divides the circumference shall be the shortest path between A and B which does not cut the $\odot$.

For, if not, if possible, let there be a path, between A and B , passing through a point P outside the $\odot$ which is as short as or shorter than any other:

and let Q, R be the points where the path through P , on opposite sides of it, meets the $\odot$ ce.



Draw a tangent to the $\odot$ at some point S in the circumference between Q and R cutting the path in T and V .

Then TV is $<$ the portion of the path TPV ;

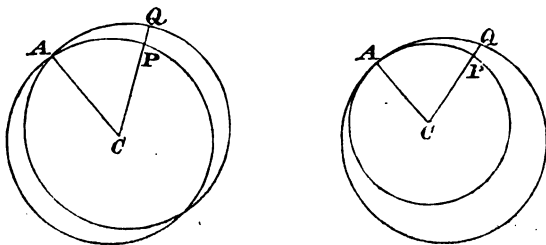
$\therefore$ a shorter path than that through P has been found; which is impossible, according to the hypothesis.

Hence it follows that the arc AB is the shortest path between A and B which does not cut the circle.

CONTACT OF CIRCLES.

PROPOSITION VII.

Circles which meet one another cannot have the same centre.



For, if possible, let the circles AP , AQ meeting in A have a common centre C .

Join AC , and draw any straight line CPQ cutting both circles.

Then $\because C$ is the centre of $\odot AP$; $\therefore CA = CP$;

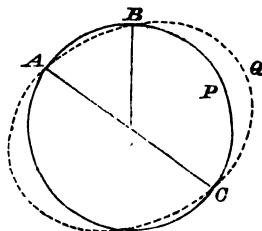
and $\because C$ is the centre of $\odot AQ$; $\therefore CA = CQ$;

$\therefore CP = CQ$, which is impossible.

Therefore, *Circles which meet, &c.*

PROPOSITION VIII.

One circle cannot meet another in more points than two.



For, if possible, let the $\odot$ s ABP , ABQ meet one another in A , B , C .

Find O the centre of the $\odot ABP$: join OA , OB , OC .

Then OA , OB , OC are = one another.

And $\therefore$ from O three equal straight lines OA , OB , OC are drawn to the circumference of $\odot ABQ$,

$\therefore O$ is the centre of $\odot ABQ$. (III. 3)

$\therefore O$ is the centre of two circles which cut one another,

but this is impossible. (III. 7)

$\therefore$ One circle, &c.

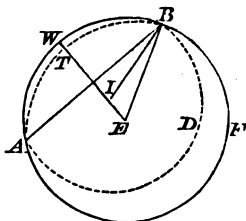
DEFINITION.

Two circles are said to **touch** one another when they are in contact, and one is entirely within or entirely without the other.

PROPOSITION IX.

One circle cannot touch another in more points than one.

First, if possible, let one $\odot ATBD$ touch another $\odot AWBF$ internally in the points A, B .



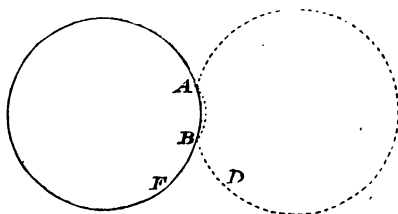
Join AB ; then the centres I, E of the interior and exterior circles respectively are in the straight line bisecting AB at right $\angle$ s. (III. 2)

Produce EI to meet the circles in T and W .

Join EB, IB .

Then EI, IB are together $> EB$; (I. 16)

$\therefore EI, IT$ are together $> EW$;
which is impossible.



Next, if possible, let one $\odot ABD$ touch another $\odot ABF$ externally in the points A, B .

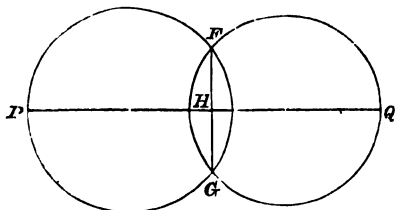
Then the straight line joining A, B must be within each of the circles; (III. 1)

which is impossible, since each falls entirely without the other.

Hence, one circle cannot touch another in more points than one. Q. E. D.

PROPOSITION X.

The straight line joining the centres of two intersecting circles bisects their common chord at right angles.



Let FG be the common chord of the intersecting circles PFG , QFG .

The straight line joining their centres shall bisect FG at right $\angle$ s.

Bisect FG in H , and draw $PHQ \perp$ to FG .

Then $\therefore$ the chord FG of $\odot PFG$ is bisected at right $\angle$ s by PQ ;

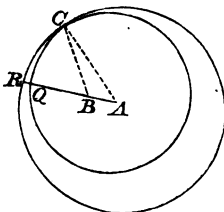
$\therefore$ centre of $\odot PFG$ is in PQ ; (III. 2)

so also in like manner is the centre of $\odot QFG$;

$\therefore$ the straight line joining the centres of the $\odot$ s PFG , QFG bisects FG at right $\angle$ s.

PROPOSITION XI.

If one circle touch another internally the straight line joining their centres shall pass through the point of contact.



Let the $\odot CQ$, whose centre is B , touch the $\odot CR$, whose centre is A , internally at C .

Then shall AB produced pass through C .

For, if not, if possible, let it fall otherwise, as $ABQR$.

Then AB, BC are together $> AC$, (I. 16)

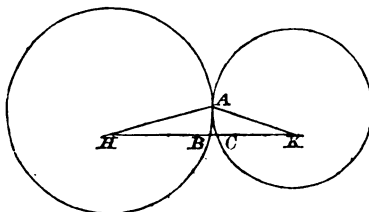
but $BC = BQ$ and $AC = AR$.

$\therefore AB, BQ$ are together $> AR$, which is impossible.

$\therefore AB$ produced must pass through C .

PROPOSITION XII.

If two circles touch externally the straight line joining their centres shall pass through the point of contact.



Let H, K be the centres of two circles AB, AC touching externally in A .

Then shall the straight line joining HK pass through A .

For, if not, if possible, let it fall otherwise, as $HBCK$.

Join HA, KA .

Then HA, AK are together $> HK$, (I. 16)

but $HA = HB$ and $AK = CK$;

$\therefore HB, CK$ are together $> HK$, which is impossible.

$\therefore$ If two circles, &c.

ANGLES IN CIRCLES.

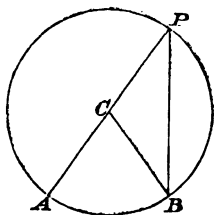
DEFINITIONS.

A **segment** of a circle is the figure contained by a straight line and the part of the circumference which it cuts off.

An angle is said to stand upon the circumference intercepted between its arms.

PROPOSITION XIII.

The angle at the centre of a circle is double of the angle at the circumference upon the same arc.



Let AB be an arc of a $\odot$ of which C is the centre,

ACB an angle upon AB at the centre,

APB an angle upon AB at the $\odot$ ce.

Then shall $\angle ACB$ be double of $\angle APB$.

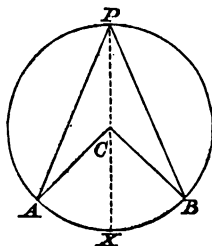
1st. Let the centre C be situated in one of the arms AP of the $\angle APB$.

Then $\because CP = CB$; $\therefore \angle CPB = \angle CBP$; (I. 2)

but $\angle ACB = \angle s\ CPB, CBP$; (I. 24)

$\therefore \angle ACB$ is double of $\angle CPB$; i.e. $\angle APB$.

2ndly. Let C lie within the $\angle APB$.



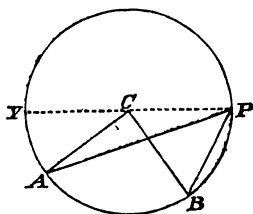
Join PC , and produce it to meet the $\odot$ in X .

Then $\angle ACX$ is double of $\angle APX$,

and $\angle BCX$ is double of $\angle BPX$;

$\therefore$ the whole $\angle ACB$ is double of $\angle APB$.

3rdly. Let C lie without the $\angle APB$.



Join PC , and produce it to meet the $\odot$ in Y .

Then $\angle BCY$ is double of $\angle BPY$,

and $\angle ACY$ is double of $\angle APY$;

$\therefore$ remaining $\angle ACB$ is double of $\angle APB$.

DEFINITION.

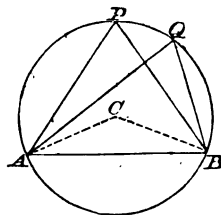
An **angle in a segment** is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.

PROPOSITION XIV.

The angles in the same segment of a circle are equal to one another.

Let $\angle APB, \angle AQB$ be angles in the same segment $APQB$.

The $\angle APB$ shall be $= \angle AQB$.



1st. Let the segment be greater than a semicircle.

Join A and B with the centre of the $\odot$.

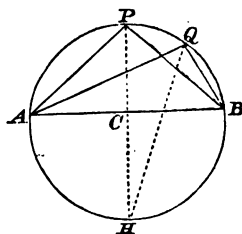
Then $\angle ACB$ is double each of the $\angle$ s APB, AQB ; (III. 13)

$\therefore \angle$ s APB, AQB are equal to one another.

Next, Let the segment not be greater than a semicircle.

Draw PH through the centre of the $\odot$:

and join QH .



Then segment $APQBH$ is $>$ a semicircle;

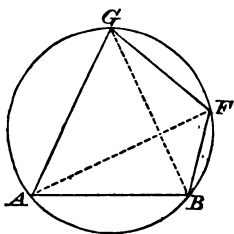
$\therefore \angle APH = \angle AQH$.

Similarly, $\angle BPH = \angle BQH$;

$\therefore$ whole $\angle APB =$ whole $\angle AQB$.

PROPOSITION XV.

The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.



Let $ABFG$ be a quadrilateral figure inscribed in the circle ABF .

Then shall the $\angle$ s ABF , AGF be together equal to two right $\angle$ s.

So also shall the $\angle$ s BAG , BFG .

Join AF , BG .

Then the $\angle AGF$ is = the $\angle$ s AGB , FGB ;

but $\angle AGB = \angle AFB$,

$\therefore$ they are in the same segment $AGFB$: (III. 14)

and $\angle FGB = \angle FAB$,

$\therefore$ they are in the same segment $FGAB$; (III. 14)

$\therefore \angle AGF =$ the $\angle$ s AFB , FAB .

$\therefore \angle AGF$ together with the $\angle ABF$ is = the $\angle$ s AFB , FAB , ABF the three $\angle$ s of $\triangle AFB$,

and $\therefore =$ two right angles. (I. 24)

Similarly it may be proved that

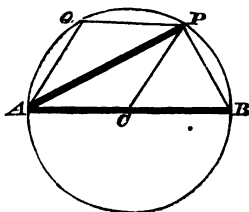
the $\angle$ s BAG , BFG are together = two right angles.

PROPOSITION XVI.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let APB be a $\odot$ of which C is the centre,

ACB a diameter, and AP a chord dividing the $\odot APB$ into the segments ABP , AQP .



Then shall the $\angle$ in the semicircle APB be a right $\angle$;

the $\angle$ in the segment ABP greater than a semicircle $<$ a right $\angle$;

and the $\angle$ in the segment AQP less than a semicircle $>$ a right $\angle$.

Join A and P with any point Q in the arc AQP ;
also join BP , CP .

Then $\because AC = CP$; $\therefore \angle APC = \angle PAC$; (I. 2)

and $\because BC = CP$; $\therefore \angle BPC = \angle PBC$; (I. 2)

$\therefore$ the whole $\angle APB =$ the $\angle$ s PAC , PBC together;

but the three $\angle$ s of $\triangle APB$ are together = two right $\angle$ s; (I. 24)

$\therefore \angle APB$ in the semicircle is a right $\angle$.

Again, $\because$ the $\angle$ s APB , ABP are together $<$ two rt. $\angle$ s, (I. 13)

and $\angle APB$ is a right $\angle$;

$\therefore \angle ABP$ in the segment ABP greater than a semicircle is $<$ a right $\angle$.

Also, $\because ABPQ$ is a quadrilateral figure in a circle,

$\therefore \angle$ s ABP , AQP are together = two right $\angle$ s, (III. 15)

but $\angle ABP$ is $<$ a right $\angle$;

$\therefore$ the $\angle AQP$ in the segment AQP less than a semicircle is $>$ a right $\angle$.

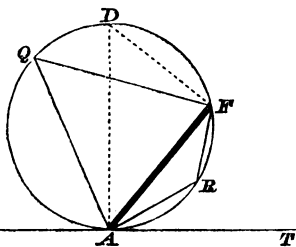
PROPOSITION XVII.

If a straight line touches a circle, and from the point of contact a straight line be drawn cutting the circle; the angles which this straight line makes with the tangent shall be equal to the angles in the alternate segments of the circle.

Let ST touch the $\odot QDF$ in the point A ; and from A let AF be drawn cutting the $\odot$:

Then shall the $\angle FAT = \angle AQF$ in the alternate segment AQF .

Also the $\angle FAS$ shall be $= \angle$ in the segment FRA .



Through A draw the diameter AD ; join DF .

Then $\angle AFD$ is a right angle; (III. 16)

$\therefore \angle s FDA, FAD$ are together $=$ a right $\angle$;
and $\therefore = \angle DAT$; (III. 2)

i.e. $= \angle s FAT, FAD$ together.

Take away the common $\angle FAD$.

$\therefore \angle FDA = \angle FAT$;
but $\angle AQF = \angle FDA$; (III. 14)
 $\therefore \angle FAT = \angle AQF$.

Again, since $\angle s FRA, AQF$ are together $=$ two right angles; (III. 15)

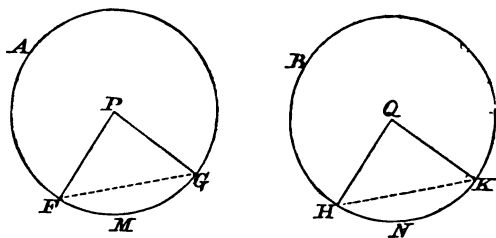
$\therefore$ they are $= \angle s FAS, FAT$; (I. 8)
but $\angle AQF = \angle FAT$;
 $\therefore \angle FRA = \angle FAS$.

LEMMA.

Circles, whose radii are equal, are equal in all respects; for, if one circle be applied to the other so that their centres coincide, then will their circumferences coincide also.

PROPOSITION XVIII.

In equal circles equal angles stand upon equal arcs.



Let AFM , BHN be equal circles; and let the $\angle$ s FPG , HQK at their centres be equal to one another.

The arc FMG shall be equal to the arc HNK .

For let the $\odot AFM$ be applied to the $\odot BHN$,
 so that P may fall on Q and PF on QH ;
 then will PG fall on QK , $\because \angle FPG = \angle HQK$;
 and the $\odot$ ce AFM will fall on the $\odot$ ce BHN ; (Lemma)
 $\therefore$ the arc FMG must coincide with arc HNK ,
 and is $\therefore$ = it.

COR. 1. The chord FG will coincide with the chord HK .

Also the tangents at F , G will coincide with the tangents at H , K .

COR. 2. *In equal circles equal chords cut off equal arcs.*

For FG being = HK , and also $PF = QH$ and $PG = QK$,
 it follows that $\angle FPG = \angle HQK$; (I. 5)

$\therefore$ arc FMG = arc HNK .

Conversely :

In equal circles equal arcs are subtended by equal angles.

In the equal circles AFM , BHN let the arcs FMG , HNK be equal.

Then shall the $\angle$ s FPG , HQK , which these arcs subtend at the centres, be equal.

For let the $\odot AFM$ be applied to the $\odot BHN$,
so that the centre P may be on Q and PF on QH ,
and $\therefore$ the $\odot$ ce AFM on the $\odot$ ce BHN .

Then G will coincide with K , $\therefore$ arc FMG is = arc HNK .

$\therefore PG$ will coincide with QK ;

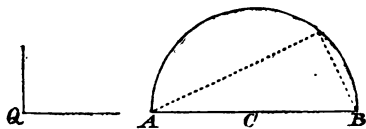
$\therefore \angle FPG$ is = $\angle HQK$.

COR. *In equal circles equal arcs are subtended by equal chords.*

For since F and G coincide with H and K ;
 $\therefore$ the chord FG will coincide with the chord HK ,
and is $\therefore$ = it.

PROBLEM D.

On a given straight line describe a segment of a circle which shall contain an angle equal to a given rectilineal angle.



Let AB be the given straight line and Q the given $\angle$.

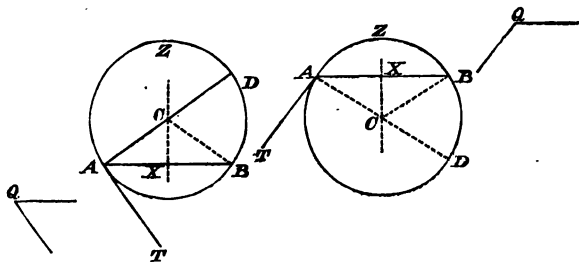
It is required to describe upon AB a segment of a circle which shall contain an $\angle = Q$.

If the $\angle Q$ is a right $\angle$,

Bisect AB in C , and with C as a centre at the distance CA describe a $\odot$.

Then segment on AB will be a semicircle,
and will $\therefore$ contain an $\angle = Q$, a right $\angle$. (III. 16)

But if the $\angle Q$ is not a right $\angle$:



Then at the point A in AB make $\angle BAT = \angle Q$.

Through A draw $AD \perp$ to AT ;

bisect AB in X ;

through X draw $XC \perp$ to AB cutting AD in C ;

with centre C at the distance CA describe a $\odot$.

Join CB .

Then $\because AX, XC$, and $\angle AXC$ are respectively BX, XC , and $\angle BXC$,

$\therefore AC = BC$ (I. 1); $\therefore$ the $\odot$ passes through B ;

and the segment AZB (which is alternate to $\angle BAT$) shall be the one required.

$\therefore AT$ is $\perp$ to ACD , $\therefore AT$ touches the $\odot$; (III. 6)

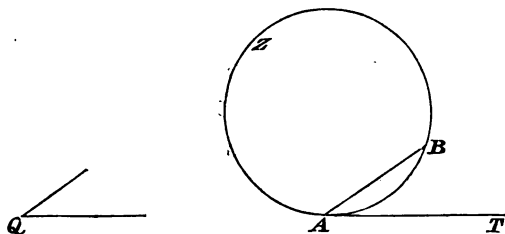
$\therefore \angle BAT = \angle$ in segment AZB ; (III. 17)

$\therefore \angle Q = \angle$ in segment AZB ;

$\therefore AZB$ is the segment required.

PROBLEM E.

From a given circle cut off a segment which shall contain an angle equal to a given rectilineal angle.



Let ABZ be the given $\odot$ and Q the given $\angle$.

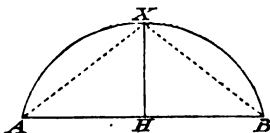
Draw any tangent AT to the $\odot$,
and at A in AT make $\angle TAB = \angle Q$.
Then shall AZB be the segment required.

For $\angle$ in segment $AZB = \angle BAT$, (III. 17)
and $\therefore = \angle Q$.

ON THE SUBDIVISION OF THE CIRCUMFERENCE OF A CIRCLE.

PROBLEM F.

To bisect a given arc of a circle.



Let AXB be the given arc of a circle,
it is required to bisect it.

Join AB and bisect it in H :
through H draw $HX \perp$ to AB ,
cutting the $\odot$ in X .
The arc AXB shall be bisected in X .

Join AX, XB .

$\therefore$ in $\triangle$ s AHX, BHX ,
 $AH = HB$ and HX is common;
also $\angle AHX = \angle BHX$;

$\therefore AX = BX$;

$\therefore \text{arc } AX = \text{arc } BX$.

(I. 1)

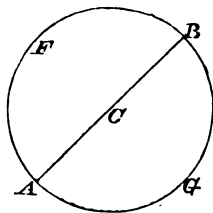
(III. 18)

PROBLEM G.

To bisect the circumference of a circle.

Through the centre C of $\odot$ AFG draw any diameter AB .

Then AB shall bisect the $\odot$ ce.



For the $\odot$ may be taken up, inverted and applied to its former position, so that AB may fall upon its former position.

Then will AGB fall upon AFB ;

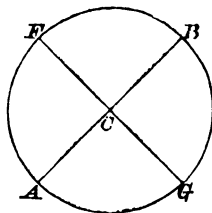
$\therefore$ arc AGB = arc AFB .

COR. *To divide the circumference of a circle into 4, 8, 16, 32, &c. equal parts.*

If through C we draw $FCG \perp$ to ACB ,

then FCG will bisect the arcs AFB , AGB .

$\therefore$ the $\odot$ ce will be divided into four equal parts.

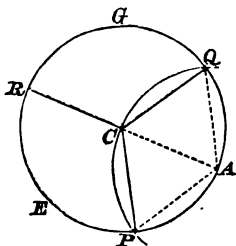


If we now bisect the $\angle$ s at the centre, each of the arcs AG , GB , BF , FA will be bisected;

and $\therefore$ the $\odot$ ce will be divided into eight equal parts; and so on.

PROBLEM H.

To divide the circumference of a circle into three equal parts.



Let C be the centre of the given circle.

With any point A in the $\bigcirc$ ce as centre and distance AC describe a $\bigcirc PCQ$;

produce AC to meet the $\bigcirc$ ce in R .

Then shall the $\bigcirc$ ce be trisected in P, Q, R .

Join CP, CQ, AP, AQ .

Then ACP, ACQ are equilateral $\triangle$ s;

$\therefore$ each of the $\angle$ s ACP, ACQ is = a third of two right $\angle$ s;

$\therefore \angle$ s PCR, QCR are each = two thirds of two right $\angle$ s,

and $\angle PCQ$ = two thirds of two right $\angle$ s;

$\therefore \angle$ s PCR, QCR, PCQ are = one another;

$\therefore$ the arcs PR, QR, PQ are = one another.

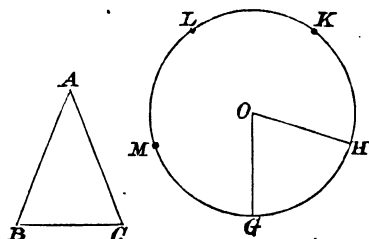
COR. Hence we may divide the circumference of a circle into 6, 12, 24, &c. equal parts.

If we produce PC, QC to meet the $\bigcirc$ ce in G, E ,
then will the $\bigcirc$ ce be divided into six equal arcs.

If we bisect these arcs it will be divided into 12 equal arcs; and so on.

PROBLEM K.

To divide the circumference of a circle into five equal parts.



Describe an isosceles $\triangle ABC$ having each of the $\angle$ s at the base double the $\angle$ at the vertex. (II. c)

At the centre of the given circle draw the $\angle GOH = \angle ABC$.

The $\bigcirc$ ce may be divided into five arcs each $= GH$.

For five $\angle$ s, each $= \angle BAC$, make up two right $\angle$ s;

$\therefore$ five $\angle$ s, each $= \angle GOH$, make up four right $\angle$ s;

$\therefore$ the $\bigcirc$ ce may be divided into five arcs each $= GH$.

COR. Hence we may divide the circumference of a circle into 10, 20, 40, &c. equal parts.

PROBLEM L.

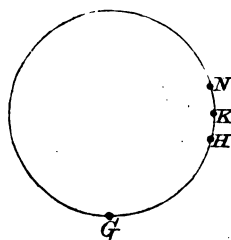
To divide the circumference of a circle into fifteen equal parts.

Cut off GN = one-third of the $\bigcirc$ ce,

and GH = one-fifth of the $\bigcirc$ ce,

bisect the arc HN in K .

Then the $\bigcirc$ ce may be divided into fifteen parts each $= HK$.



For of such parts, as the $\bigcirc$ ce contains fifteen, GN contains five, and GH contains three.

$\therefore$ HN contains two;

$\therefore$ each of the parts is $= HK$.

COR. Hence the $\bigcirc$ ce of a $\odot$ may be divided into 30, 60, &c. equal parts.

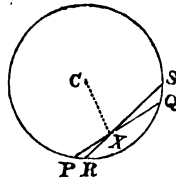
SUPPLEMENT TO BOOK III.

THEOREM (a).

If in a circle two chords which do not both pass through the centre, cut one another, they do not bisect each other.

If one of them pass through the centre it is plain that it cannot be bisected by the other which does not pass through the centre.

But if neither of them pass through the centre, as PXQ, RXS , find the centre C , and join CX .



Then $\because CX$ bisects PQ , $\therefore CX$ is $\perp$ to PQ ; (III. 2)

and $\because CX$ bisects RS , $\therefore CX$ is $\perp$ to RS ; (III. 2)

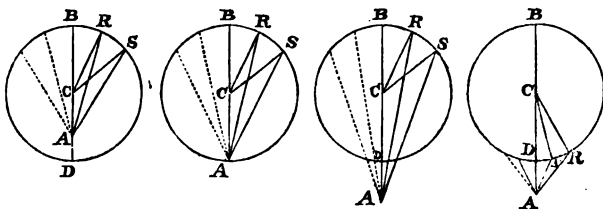
$\therefore \angle CXP$ and CXR are right $\angle$ s,

and are $\therefore =$ one another; which is impossible.

$\therefore$ *If in a circle two chords &c.*

THEOREM (b).

Of all straight lines which can be drawn from a given point, which is not the centre, to the circumference, the greatest is that which passes through the centre. And of the others, on the same side of that through the centre, the one drawn to a point on the circumference nearer to the extremity of that line is the greater.



Let A be the given point and C the centre of the given $\odot$.

Through A draw ACB and also any other straight lines AR, AS to the $\odot$.

Then shall ACB be $> AR$ and $AR > AS$.

Join CR, CS .

Then AC, CR are together $> AR$; (I. 16)

$\therefore AC, CB$ are together $> AR$,

i. e. AB is $> AR$.

Again $\because AC, CR$ are respectively $= AC, CS$,

and $\angle ACR$ is $> \angle ACS$,

$\therefore AR$ is $> AS$. (I. 18)

COR. 1. *If the given point be not on the $\odot$ ce, then that straight line, which if produced would pass through the centre, is the shortest.*

For if the given point A be within the $\odot$,

then CA, AS are together $> CS$,

and $\therefore > CD$;

$\therefore AS$ is $> AD$;

and if the given point A be without the $\odot$,

then CS, AS are together $> CA$;

$\therefore AS$ is $> AD$.

COR. 2. *To every one of the straight lines, on one side of that through the centre, there can be drawn one, and only one, equal to it on the other side.*

For the figure may be turned about the line through the centre till the $\odot$ ce on one side of that line shall fall on the former position of the $\odot$ ce on the other side;

and $\therefore$ to all the straight lines AR, AS &c.

on one side of the line through the centre

there will be equal straight lines on the opposite side.

Also there cannot be two equal straight lines on the same side.

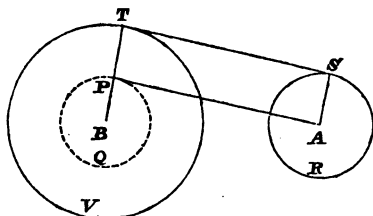
COR. 3. If the distance between the centres of two $\odot$ s be $>$ the sum or $<$ the difference of their radii, the $\odot$ s will not meet.

If the distance be $=$ the sum or difference, the $\odot$ s will meet in one point only.

If the distance be $<$ the sum and $>$ the difference
the $\odot$ s will meet in two points.

PROBLEM (c).

Draw a common tangent to two given circles.



If the $\odot$ s are equal, the construction is easy. But if not, let A be the centre of the smaller $\odot$ and B of the greater.

With centre B and radius = the difference between the radii of the given $\odot$ s describe a $\odot PQ$.

From A draw a tangent AP to $\odot PQ$.

Join BP and produce it to T ; draw $AS \parallel$ to BT ; join ST .

Then ST shall be the tangent required.

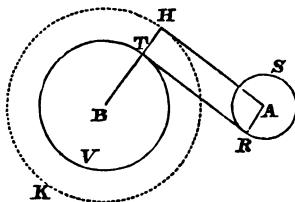
$\therefore AS$ is = and $\parallel$ to PT , $\therefore AT$ is a $\square$, (I. 23)

and $\therefore TPA$ is a right $\angle$, $\therefore \square AT$ is a rectangle; (I. 27)

$\therefore \angle$ s at S and T are right $\angle$ s;

$\therefore ST$ touches the $\odot$ s RS, TV . (III. 6)

We may also draw a common tangent to RS , TV thus:
with centre B describe a $\odot HK$ whose radius is = the sum
of the radii of the given circles.



From A draw a tangent AH to $\odot HK$.

Join BH cutting TV in T ;

draw $AR \parallel$ to BH ; join RT .

Then RT shall be a common tangent to RS , TV .

For AR is = and $\parallel$ to HT , $\therefore HR$ is a $\square$,

and $\angle AHT$ is a right $\angle$; (III. 6)

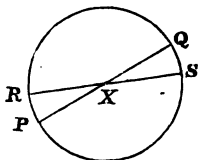
$\therefore HR$ is a rectangle, and $\therefore \angle$ s at T , R are right $\angle$ s;

$\therefore RT$ touches the given $\odot$ s. (III. 6)

Proofs of the two following theorems depending on
the first three Books are here given; but more elegant
demonstrations will be found in Book VI.

THEOREM (d).

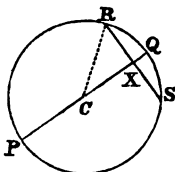
If two chords of a circle intersect, the rectangles contained by their segments are equal.



Let the two chords PQ , RS of the $\odot PRQ$ intersect in X ;
then shall rectangle (PX, XQ) be = rectangle (RX, XS) .

If PQ , RS intersect in the centre of the $\odot$: then PX ,
 XQ , RX , XS are all equal;
and $\therefore$ rectangle (PX, XQ) is = rectangle (RX, XS) .

If one of them, as PQ , pass through the centre C , and



cut the other RS which does not pass through the centre at
right angles, and accordingly bisect RS ,
join CR .

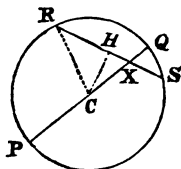
Then $\therefore PX$ is = the sum and XQ = the difference of
 CX and CQ ;

$\therefore$ rectangle (PX, XQ) + square on CX = square on CQ ,
(II. 2)

and $\therefore$ = square on CR , and $\therefore$ = squares on CX, XR ; (I. 35)

$\therefore$ rectangle (PX, XQ) = square on XR , i.e. = rectangle
 (RX, XS) .

If one of them, as PQ , pass through the centre and cut



the other RS which does not pass through the centre, but not at right angles,

from C let fall $CH \perp$ to RS ,
and join CR .

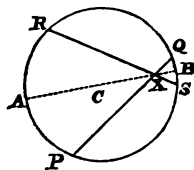
Then rectangle $(PX, XQ) + \text{square on } CX = \text{square on } CQ;$ (II. 2)

$\therefore$ rectangle $(PX, XQ) + \text{squares on } CH, HX = \text{square on } CR;$ and $\therefore = \text{squares on } CH, HR;$ (I. 35)

$\therefore$ rectangle $(PX, XQ) + \text{square on } HX = \text{square on } HR;$
but rectangle $(RX, XS) + \text{square on } HX = \text{square on } HR,$ (II. 2)

$\therefore$ rectangle $(PX, XQ) = \text{rectangle } (RX, XS).$

Lastly, if neither of them pass through the centre,



through X draw the diameter $ACXB$.

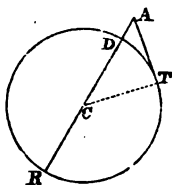
Then rectangle (PX, XQ) is = rectangle $(AX, XB),$

and $\therefore = \text{rectangle } RX, XS.$

THEOREM (e).

If from a point without a circle a secant and a tangent be drawn, the rectangle contained by the parts of the secant between the given point and the circle is equal to the square on the tangent.

From a point A without the $\odot DTR$ let the tangent AT and the secant ADR be drawn.



Then shall rectangle $(AR, AD) = \text{square on } AT$.

1st. If the secant passes through the centre C : join CT .

Then $\therefore AR$ is the sum, and AD = the difference of CD and CA ;

$\therefore \text{rectangle}(AR, AD) + \text{square on } CD = \text{square on } CA$, (II. 2)
and $\therefore = \text{squares on } CT, AT$; (III. 6), (I. 35)

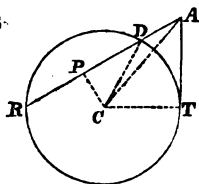
but square on $CD = \text{square on } CT$;

$\therefore \text{rectangle}(AR, AD) = \text{square on } AT$.

Next: If the secant does not pass through the centre,

let fall $CP \perp$ on ADR .

and join CD, CT, CA .



Then rectangle $(AR, AD) + \text{square on } DP = \text{square on } AP$; (II. 2)

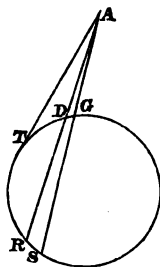
$\therefore \text{rectangle}(AR, AD) + \text{squares on } DP, CP = \text{squares on } AP, CP$;

$\therefore \text{rectangle}(AR, AD) + \text{square on } CD = \text{square on } AC$,
and $\therefore = \text{squares on } CT, AT$; (I. 35), (III. 6)

but square on $CD = \text{square on } CT$;

$\therefore \text{rectangle}(AR, AD) = \text{square on } AT$.

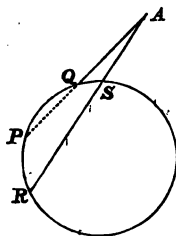
COROLLARY. *If from any point A without a $\odot$ TDG , two straight lines ADR , AGS be drawn cutting the $\odot$, then rectangle (AD, AR) is = rectangle (AG, AS) .*



For each of them is = the square on the tangent AT drawn from A to the $\odot$.

Conversely :

If from a point A without a $\odot$ two straight lines are drawn, one of them ASR cutting the $\odot$, and the other AQ meeting it, and rectangle AS, AR = square on AQ : then shall AQ touch the $\odot$.



For if not, if possible, let it cut the $\odot$ in some other point P .

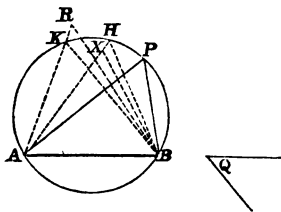
Then rectangle AQ, AP = rectangle AS, AR ; and

$\therefore$ = square on AQ , which is impossible;

$\therefore AQ$ does touch the $\odot$.

PROBLEM (f).

Find the locus of the vertices of all triangles, on the same base and on the same side of it, having a given vertical angle.



Let AB be the given base, and Q the given angle.

On AB describe a segment of a $\odot$ capable of containing an angle $= Q$.

The arc of this segment shall be the locus required.

For join A, B with any point P in the arc :

then the $\triangle APB$ has its vertical angle $= Q$;

so every $\triangle$ whose vertex is in the arc APB ,
having AB for its base, has its vertical $\angle = Q$;

and conversely, every $\triangle$ having its vertical $\angle = Q$, and AB
as its base, and situated on the same side of AB as the
segment APB , will have its vertex in the arc APB .

For if not, if possible let AXB be such a $\triangle$ having its
vertex within the segment ; produce AX to H , join BH .

Then $\angle AXB > \angle AHB$, and $\therefore > Q$, but it is also $=$ it,
which is impossible.

Again, if possible, let ABR be a $\triangle$ on base AB , having
its vertical $\angle = Q$,

the vertex R being without the segment ; join BK .

Then $\angle ARB$ is $< \angle AKB$, and $\therefore < Q$; but it is
also $=$ it ; which is impossible.

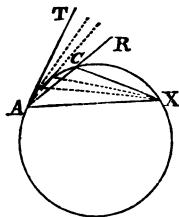
Hence the locus required is the arc APB .

DEFINITION.

The angle of a segment is that which is contained by the base of the segment and the tangent at one extremity.

THEOREM (g).

Assuming the tangent to be the limiting position of the secant, prove that the angle of a segment of a circle and the angle in it are together equal to two right angles.



Let AT be the tangent at one extremity of the base AX of the segment ACX .

Then shall the $\angle XAT$ and the $\angle$ in the segment ACX be together equal to two right angles.

Through A draw a secant ACR ,
and join XC .

Then angles XAC , AXC , and the angle in the segment ACX , are together equal to two right angles. (I. 24), (III. 14)

Now let the secant ACR be turned about A towards AT , and ultimately coincide with it, then will C move along the arc towards A , and ultimately coincide with it ;
also $\angle AXC$ will continually diminish,
and ultimately vanish.

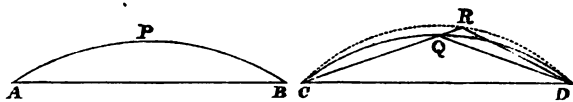
Hence the $\angle XAT$ and the $\angle$ in the segment ACX are together equal to two right angles.

DEFINITION.

Similar segments are those which contain equal angles.

THEOREM (h).

Similar segments of circles having equal chords are equal in all respects.



Let APB , CQD be similar segments of circles having equal chords AB , CD .

Then shall the segments APB , CQD be equal in all respects.

For let the segment APB be applied to the segment CQD so that the chord AB may be on CD ;
then will the arc APB be on the arc CQD ;
for if not, if possible, let it fall otherwise.

Then since these arcs cannot meet in more points than two, one of them, as CQD , must lie within the other. (III. 8)

Draw the straight line CQR cutting the arcs in Q and R ;
join DQ , DR .

Now $\therefore$ the segments CQD , CRD are similar

$\therefore \angle CQD = \angle CRD$,

which is impossible; (I. 13)

$\therefore$ the arc APB will fall on the arc CQD ;

$\therefore$ the segments APB , CQD are equal in all respects.

COR. Upon the same base and on the same side of it there cannot be two similar segments of circles not coinciding with one another.

BOOK IV.

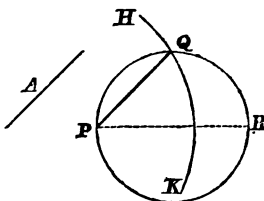
INSCRIBED AND CIRCUMSCRIBED FIGURES.

TRIANGLES AND CIRCLES.

PROBLEM I.

In a given circle draw a chord equal to a given straight line which is not greater than the diameter of the circle.

Let PQR be the given circle, and A the given straight line, not greater than the diameter of the circle.



It is required to draw a chord in the circle PQR equal to A .

With any point P , in the circumference PQR , as centre and radius equal to A describe a circle HQK , which must meet the diameter drawn through P , and will therefore meet the circle in some point Q ;

join PQ .

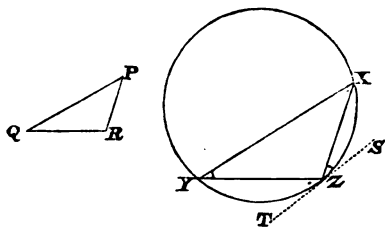
Then PQ is a chord of the circle PQR ,
and is equal to A .

DEFINITION.

A rectilineal figure is said to be **inscribed in** a circle when all the angular points of the rectilineal figure are upon the circumference of the circle.

PROBLEM 2.

In a given circle to inscribe a triangle equiangular to a given triangle.



Let it be required to inscribe in the $\odot XYZ$ a $\triangle$ equiangular to the $\triangle PQR$.

Draw a tangent SZT to the $\odot XYZ$:
 at Z in ZS make $\angle SZX = \angle PQR$,
 and at Z in ZT make $\angle TZY = \angle QPR$:
 join XY : then XYZ shall be the $\triangle$ required.

For the $\angle XYZ$ is = the $\angle SZX$, (III. 17)

which was made = $\angle PQR$;

and $\angle YXZ$ is = $\angle TZY$, (III. 17)

which was made = $\angle QPR$;

$\therefore$ the remaining $\angle YZX$ = the remaining $\angle QRP$. (I. 25)

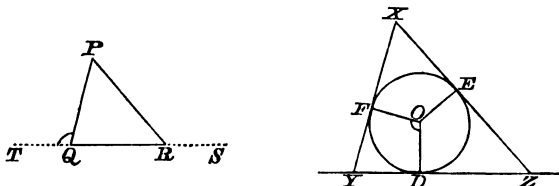
Hence $\triangle XYZ$ is equiangular to $\triangle PQR$.

DEFINITION.

A rectilinear figure is said to be **described about** a circle, when each side of the rectilinear figure touches the circumference of the circle.

PROBLEM 3.

About a given circle describe a triangle equiangular to a given triangle.



Let it be required to describe about the $\odot DEF$ a $\triangle$ equiangular to $\triangle PQR$.

Produce QR both ways to S, T .

Draw any radius OD and make $\angle DOE = \angle PRS$,
and $\angle DOF = \angle PQT$.

At D, E, F draw tangents to the $\odot$ intersecting in X, Y, Z .
 XYZ shall be the $\triangle$ required.

$\therefore$ the four $\angle$ s of a quadrilateral figure are together =
four right $\angle$ s, (I. 30)

and $\angle$ s ODZ, OEZ are right $\angle$ s; (III. 6)

$\therefore \angle$ s DOE, DZE are together = 2 right $\angle$ s,
and $\therefore = \angle$ s PRS, PRQ ; (I. 9)

but $\angle DOE = \angle PRS$; $\therefore \angle DZE = \angle PRQ$.

Similarly it may be proved that the $\angle$ at $Y = \angle PQR$;

$\therefore$ the remaining $\angle$ at $X = \angle QPR$; (I. 25)

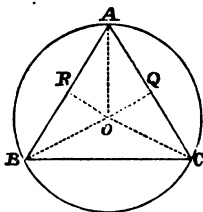
$\therefore$ therefore the $\triangle XYZ$ is equiangular to the $\triangle PQR$,
and it has been described about the $\odot DEF$.

DEFINITION.

A circle is said to be **described about** a rectilinear figure when the circumference of the circle passes through all the angular points of the rectilinear figure.

PROBLEM 4.

To describe a circle about a given triangle.



Let ABC be the given Δ .

It is required to describe a $\odot$ about the ΔABC .

Bisect AC in Q and AB in R .

Through Q and R draw straight lines $\perp$ to AC and AB respectively, which will intersect in some point O ; otherwise they would be $\parallel$, and $\therefore AC, AB$ would be $\parallel$, which is impossible.

Join OA , and also OB, OC , unless O be in BC .

With O as centre and OA as radius describe a $\odot$;
it shall be the one required.

$\therefore AR, RO$ and $\angle ARO$ are respectively $= BR, RO$,
and $\angle BRO$,

$$\therefore OA = OB. \quad (1. 1)$$

Similarly

$$OA = OC.$$

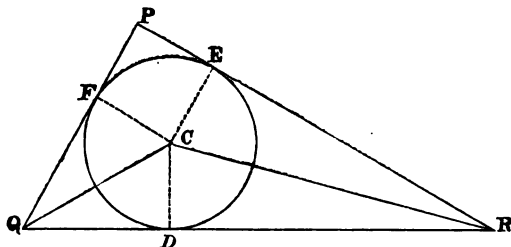
$\therefore$ the $\odot$ passes through B and C , and is $\therefore$ described about the ΔABC .

DEFINITION.

A circle is said to be **inscribed in** a rectilineal figure when the circumference of the circle touches each side of the figure.

PROBLEM 5.

To inscribe a circle in a given triangle.



Let PQR be the given Δ ;
it is required to inscribe a $\odot$ within it.

Bisect $\angle s$ PQR , PRQ by QC , RC , intersecting in C .

From C let fall CD , CE , $CF \perp$ to QR , RP , PQ .

Then the $\odot$ described with centre C at the distance CD shall be the $\odot$ required.

$\therefore$ the $\angle s$ QDC , DQC , and the side QC are respectively = $\angle s$ QFC , FQC , and the side QC ,

$$\therefore DC = FC. \quad (I. 25)$$

Similarly it may be proved that $DC = EC$.

$\therefore$ the $\odot$ described with centre C at the distance CD will pass through E and F .

Also the sides of the ΔPQR will touch the $\odot$ because the $\angle s$ at D , E , F are right $\angle s$. (III. 6)

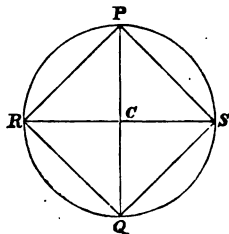
$\therefore$ a $\odot$ has been inscribed in the ΔPQR .

C. G.

SQUARES AND CIRCLES.

PROBLEM 6.

To inscribe a square in a given circle.



In the given $\odot$ draw two diameters PCQ , RCS at right $\angle$ s to one another.

Join PS , SQ , QR , RP .

Then $PSQR$ shall be the square required.

$\therefore$ the $\angle$ s at C are right $\angle$ s ; $\therefore$ they are = one another ;
and $\therefore$ the chords PS , SQ , QR , RP on which they stand are equal ; (III. 18)

$\therefore PSQR$ is equilateral.

Again, $\therefore \angle RPS$ is in a semicircle ; $\therefore$ it is a right $\angle$. (III. 16)

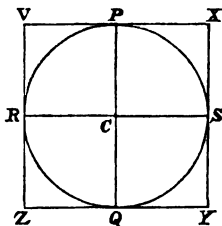
Similarly the other $\angle$ s of $PSQR$ are right $\angle$ s ;

$\therefore PSQR$ is rectangular, and it has been proved to be equilateral ;

$\therefore PSQR$ is a square and it is inscribed in the given $\odot$.

PROBLEM 7.

To describe a square about a given circle.



In the given $\odot$ draw two diameters PCQ , RCS at right $\angle$ s to one another:

at P , S , Q , R draw tangents intersecting in V , X , Y , Z .

$VXYZ$ shall be the square required.

$\therefore$ the $\angle$ s at P and C are right $\angle$ s; (III. 6)

$\therefore VPX$ is $\parallel$ to RCS . (I. 20)

Similarly ZQY is $\parallel$ to RCS ;

$\therefore VPX$ is $\parallel$ to ZQY . (I. 22)

So VRZ , $XS Y$ are each $\parallel$ to PCQ and are $\therefore \parallel$ to one another.

Then VX and ZY are each $= RS$; $\therefore VS$, SZ are $\square$ s:

also VZ , XY are each $= PQ$; $\therefore VQ$, PY are $\square$ s:

but $RS = PQ$; $\therefore VX$, ZY , VZ , XY are $=$ one another;

$\therefore VY$ is equilateral.

Also angle at $X = \angle PCS$; $\therefore PXSC$ is a $\square$. (I. 26)

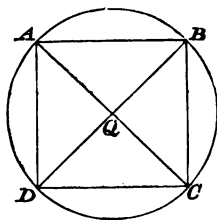
$\therefore \angle$ at X is a right $\angle$; so $\angle$ s at Y , Z , V are right $\angle$ s;

$\therefore VY$ is rectangular.

Hence VY is a square and it is described about the $\odot$.

PROBLEM 8.

To describe a circle about a given square.



Let $ABCD$ be the given square :

it is required to describe a $\odot$ about it.

Draw the diagonals AC , BD intersecting in Q .

With centre Q at the distance QA describe a $\odot$.

It shall be the one required.

$$\therefore AB = BC; \therefore \angle BAC = \angle BCA; \quad (1. 2)$$

and since $\angle ABC$ is a right $\angle$;

$$\therefore \angle s \ BAC, \ BCA \text{ are each of them} = \text{half a right } \angle. \quad (1. 24)$$

Similarly $\angle ABD = \text{half a right } \angle$, and $\therefore = \angle BAC$;

$$\therefore AQ = BQ. \quad (1. 4)$$

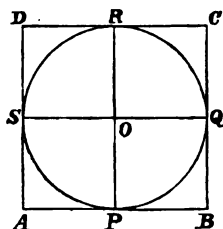
Similarly $BQ = CQ = DQ$;

$\therefore$ the $\odot$ passes through B, C, D ;

and is $\therefore$ described about the square $ABCD$.

PROBLEM 9.

To inscribe a circle in a given square.



Let $ABCD$ be the given square :
it is required to inscribe a circle in $ABCD$.

Bisect AB, BC in P, Q .

Through P, Q draw $PR, QS \parallel$ to BC, AB , respectively.

With O as centre and distance OP describe a $\odot$:

it shall be the one required.

$\therefore$ the opposite sides of $\square$ s are = one another ; (I. 26)

$\therefore SO = AP, OQ = PB, PO = QB$, and $OR = QC$;

$\therefore SO, OQ, PO, OR$ are all = one another ;

$\therefore$ the $\odot$ will pass through Q, R, S :

also $\because \angle SAP$ is a right $\angle \therefore \angle OPA$ is a right $\angle$. (I. 21)

Similarly the $\angle$ s at Q, R, S are right $\angle$ s ;

$\therefore$ the sides of the square touch the $\odot$. (III. 6)

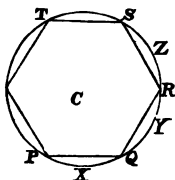
$\therefore$ the $\odot$ is inscribed in the square.

We now proceed to solve the general problems of inscribed and circumscribed regular figures, the constructions in the case of squares having been given separately on account of their simplicity.

REGULAR POLYGONS AND CIRCLES.

PROBLEM 10.

To inscribe a regular polygon having a given number of sides in a circle.



This problem can be solved if the $\odot$ ce of a $\odot$ can be divided into the given number of equal arcs.

Let the $\odot$ ce of the $\odot$ PQR be divided into a number of equal arcs PQ , QR , RS &c.

Draw the chords PQ , QR , RS , &c.

Then shall $PQRS$ be a regular polygon.

For if the figure be turned about its centre C ,
so that P may coincide with the former position of Q ,
and $\therefore Q$ with that of R , R with that of S and so on,
 $\therefore$ the arcs PXQ , QYR , RZS , &c. are all equal;
then PQ will coincide with that of QR , QR with that
of RS , &c.;

$\therefore PQ = QR = RS$, &c.; $\therefore$ the figure is equilateral;
also the arms of the $\angle PQR$ will coincide with those
of the $\angle QRS$.

Hence $\angle PQR = \angle QRS = \angle RST$, &c.; $\therefore$ the figure is equiangular;

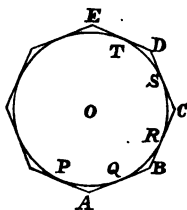
$\therefore PQRS$ is a regular figure, and it is inscribed in the given $\odot$.

COROLLARY. *Hence we may inscribe in a circle regular figures of 3, 6, 12, &c.; 4, 8, 16, &c.; 5, 10, 20, &c.; 15, 30, &c. sides.* •

(III. G. H. K.)

PROBLEM II.

To describe about a given circle a regular polygon having a given number of sides.



This problem can be solved if the $\odot$ ce of a $\odot$ can be divided into the given number of equal arcs.

Let the $\odot$ ce of the given $\odot$ be divided into a number of equal arcs PQ, QR, RS &c.;

and let tangents be drawn at P, Q, R, S &c. intersecting in A, B, C &c.

Then shall $ABCD$ be a regular polygon circumscribing the given $\odot$.

For if the figure be turned about its centre O ,
so that P may coincide with the former position of Q :
then will Q, R, S &c. coincide with the former positions of
 R, S, T &c.;

and the tangents at P, Q, R, S &c. with those of the
tangents at Q, R, S, T &c.

Hence the polygon will coincide with its former position ;

$\therefore AB = BC = CD = DE$ &c. ;

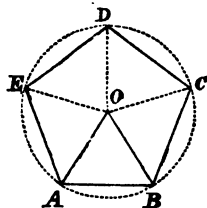
also $\angle ABC = \angle BCD = \angle CDE$, &c.,

$\therefore$ the polygon is equilateral and also equiangular, i.e. it is
a regular polygon and it is circumscribed about the given
circle.

COROLLARY. *Hence we may describe about a circle regular
figures of 3, 4, 5, 6, 8 &c. sides.* (III. G. H. K.)

PROBLEM 12.

To describe a circle about a given regular figure.



Let $ABCDE$ be the given regular figure.

It is required to describe a $\odot$ about it.

Bisect $\angle s$ EAB, ABC by AO, BO intersecting in O ,
and with centre O at distance OA describe a $\odot$.

This shall be the $\odot$ required.

Join OC, OD , &c.

Then $\because EA, AO$ and $\angle EAO$ are respectively
 $= BA, AO$ and $\angle BAO$;

$\therefore \angle OEA = \angle OBA$ (I. 1) = half of one of the $\angle s$ of the
 regular polygon;

$\therefore \angle AED$ is bisected.

Similarly it may be proved that all the $\angle s$ of the polygon
 are bisected.

Then $\because \angle OAB = \angle OBA$, $\therefore OA = OB$: (I. 4)

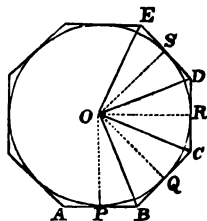
similarly $OB = OC$, &c.;

$\therefore$ the circle passes through B, C, D , &c.,

and is $\therefore$ described about the given polygon.

PROBLEM 13.

To inscribe a circle in a given regular figure.



Let $ABCD$ &c. be the given regular polygon.

It is required to inscribe a circle within it.

Bisect the $\angle$ s ABC , BCD by BO , CO intersecting in O .

From O let fall $OP \perp$ on AB .

Then the $\odot$ described with centre O at the distance OP shall be the one required.

Join OD , OE , &c., and draw OQ , OR , &c. $\perp$ to BC , CD , &c.

Then it may be shewn as in the previous problem that the $\angle$ s of the polygon are bisected by OD , OE , &c.

Also $\therefore \angle$ s OPB , OBP and OB are respectively $= \angle$ s OQB , OBQ and OB ;

$$\therefore OP = OQ. \quad (\text{i. } 25)$$

Similarly $OQ = OR = OS$ &c.

$\therefore$ the $\odot$ will pass through Q , R , S &c.;

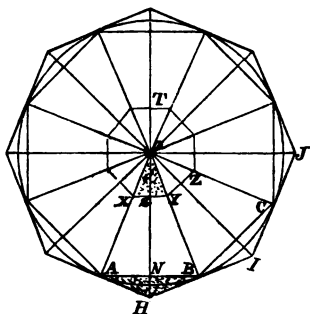
and it will touch AB , BC , CD , &c., $\therefore$ the $\angle$ s at P , Q , R , &c. are right $\angle$ s; (III. 6)

$\therefore$ it is inscribed in the given polygon.

AREAS AND PERIMETERS OF INSCRIBED AND CIRCUMSCRIBED POLYGONS.

PROBLEM 14.

To find the difference between the areas of two regular polygons having the same number of sides, one inscribed in, and the other circumscribed about a given circle.



Let $ABC\dots$ be a regular polygon inscribed in a circle and $HIJ\dots$ one of the same number of sides circumscribed about it,

formed by drawing tangents to the circle at $A, B, C\dots$

Join $OA, OB, OC, \&c. OH, OI, OJ, \&c.$

From OH cut off $OS = AN$, and draw $XSY \parallel$ to AB .

Then $\triangle OSX$ is $= \triangle ANH$ in all respects, (1. 25)

so $\triangle OSY = \triangle BNH$;

$\therefore \triangle OXY = \triangle AHB$;

cut off $OZ, \&c. = OX$ or OY .

Then $\triangle s OXY, OYZ, \&c.$ are all equal to one another,

as are also $AHB, BIC, \&c.$

Hence the difference between the polygons $ABC\dots$, and

$HIJ\dots$ is equal to polygon $XYZ\dots$

COR. 1. If the polygons are formed by the continual bisections of the circumference, then the polygon XYZ is less than a square whose side is ST (which is $= AB$).

Hence the difference between the polygons ABC, HIJ , and $\therefore$ the difference between either polygon and the circle ABC , is less than the square on a side of the inscribed polygon.

COR. 2. By continual bisections, the circumference of a circle can be divided into arcs each less than any assignable magnitude;

and $\therefore$ the chords will each be less than any assignable magnitude;

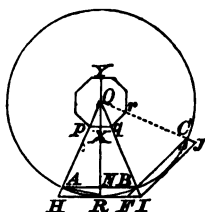
$\therefore$ the $\odot$ ce may be subdivided till the squares on the subtended chords shall each be less than any assignable area.

$\therefore$ a regular polygon can be inscribed in a circle whose area shall differ from that of the circle by less than any assignable area:

so also can one be circumscribed about it, &c.

PROBLEM 15.

To find the difference between the perimeters of two regular polygons of the same number of sides, one inscribed in, and the other circumscribed about a given circle.



Let $ABC...$ be a regular polygon inscribed in the circle ABC , and $HIJ...$ a regular polygon of the same number of sides circumscribed about it, formed by drawing tangents to the $\odot \parallel$ to the sides of $ABC...$

Draw OAH , OBI , also $BF \parallel$ to AH . Then $FI =$ difference between AB and HI .

Cut off Op , Oq , Or , &c. equal to AH or BI : join pq , qr , &c. then $pq = FI$. (1. 1)

Now complete the polygon pqr .

Then the perimeter of this polygon is equal to the difference between the perimeters of the polygons ABC , HIJ .

COR. 1. If the polygons are formed by continual bisections of the circumference, then the perimeter of the polygon pqr is less than that of a square whose side is XY .

Now XY is double of OX , and $\therefore$ double of NR ;
but NR is $< AN$, for $\angle NAR$ is $< \angle OAR$ and $\therefore < \angle ARN$;

$\therefore XY$ is $< AB$;

$\therefore$ the difference between the perimeters of the given polygons,

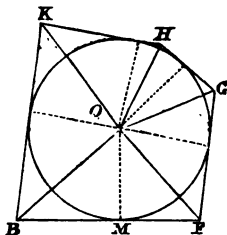
and $\therefore$ between either of them and the given circle is less than the perimeter of the square on a side of the inscribed polygon. (III. c)

COR. 2. *Hence a polygon can be inscribed in a circle whose perimeter shall differ from that of the circumference of the circle by less than any assignable magnitude.*

So also can a polygon be circumscribed about a circle, &c.

PROBLEM 16.

To find the area of a polygon circumscribed about a circle.



Let $BFGHK$ be a polygon circumscribed about a circle.
Join the centre Q with the angular points of the polygon,
and also with the points of contact.

Then the area of $\triangle BFQ$ is $= \frac{1}{2}$ rectangle (radius QM ,
and BF), (I. 32)

so area of $\triangle FGQ$ is $= \frac{1}{2}$ rectangle (radius, and FG),
&c. &c.

$\therefore$ area of polygon $= \frac{1}{2}$ sum of the rectangles contained
by (radius, and BF), (radius, and FG) &c.

$= \frac{1}{2}$ rectangle (radius, and a straight line = the perimeter).
(II. 1)

COR. Now let the points of contact of the sides of
the polygon be determined by continual bisections of the
circumference, then ultimately the area of the polygon will
differ from that of the circle by less than any assignable
magnitude, as also will the perimeter of the polygon from
the circumference of the $\odot$.

Hence the area of the circle is $= \frac{1}{2}$ rectangle (radius,
and a straight line = the circumference).

BOOK V.

RATIO AND PROPORTION.

INTRODUCTORY.

Two magnitudes of the same kind may be compared by considering how many times one contains the other.

If one does not contain the other a certain number of times exactly, it may happen that each contains some other magnitude a certain number of times, and then the two given magnitudes may be compared.

If each of two magnitudes does not contain some other magnitude a certain number of times, they are called incommensurable.

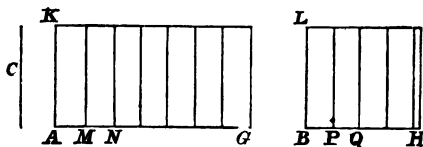
In order to institute a comparison, one of them may be divided into a number of equal parts as small as we please, and parts equal to these may be taken from the other till there remains less than one of these parts, and therefore all the parts together thus taken away differ from the whole magnitude by less than one of these parts. Hence the former magnitude may be compared with one which differs from the latter by less than any assignable quantity.

DEFINITION.

The first of four magnitudes is said to have **the same ratio** to the second which the third has to the fourth when, if the first be divided into any number whatever of equal parts and the third be divided into the same number of equal parts, the second contains the same integral number of the former parts as the fourth does of the latter.

PROPOSITION I.

Straight lines have to one another the same ratio as rectangles of equal altitude described upon them.



Let AG , BH be the given straight lines,
and C the given altitude.

Then shall AG have to BH the same ratio as rectangle (AG, C) to rectangle (BH, C) .

On AG , BH construct rectangles having altitudes each equal to C .

If now AG be divided into any number whatever of equal parts AM , MN , &c.,

and lines be drawn through M , N , &c. $\parallel$ to AK :

then KG will be divided into the same number of equal parts as AG .

If also from BH parts BP , PQ , &c., each equal to AM be cut off until there is no remainder or a remainder less than AM ,

and lines be drawn through P , Q &c. $\parallel$ to LB :

then LH will contain the same integral number of parts, each equal to KM as BH does parts each equal to AM ;

$\therefore AG$ has the same ratio to BH as KG has to LH ,

i.e. as rectangle (AG, C) has to rectangle (BH, C) .

COR. In a similar manner it may be proved that straight lines have to one another the same ratio as $\square$ s or $\triangle$ s of equal altitude described upon them.

DEFINITION.

Magnitudes which have the same ratio are called **proportionals**.

The fact of four magnitudes being proportionals is expressed thus:

The first *is to* the second *as* the third *is to* the fourth,
and written thus—

the first : the second *as* the third : the fourth.

If two magnitudes are equal one will contain the same integral number of parts, each equal to any third magnitude, as the other. Hence

PROPOSITION II.

Proportions hold good if for the magnitudes involved others equal to them are substituted.

For the test expressed by the definition is satisfied.

PROPOSITION III.

If four magnitudes are proportional then any others which have the same ratio as the first and second are proportional to any others which have the same ratio as the third and fourth.

For the test expressed by the definition is satisfied.

PROPOSITION IV.

If a magnitude have the same ratio to each of two others then these two must be equal.

For if A be the former and B , C the two latter magnitudes, so that

$$A : B \text{ as } A : C,$$

then B must be equal to C .

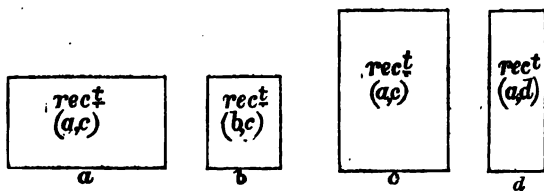
Otherwise A might be divided into a number of equal parts, each less than the difference between B and C ;

and B and C could not contain the same integral number of those parts; but they do (by the definition);

$\therefore B$ is equal to C .

PROPOSITION V.

If four straight lines are proportional the rectangle contained by the extremes is equal to that contained by the means.



Let the four straight lines a, b, c, d be proportional.

Then rect. (a, d) is = rect. (b, c) .

On a, b construct rectangles, the altitude of each being equal to c , and on c, d construct rectangles, the altitude of each being equal to a .

By hypothesis $a : b$ as $c : d$.

But $a : b$ as rect. $(a, c) : \text{rect. } (b, c)$; (v. 1)

and $c : d$ as rect. $(a, c) : \text{rect. } (a, d)$; (v. 1)

$\therefore \text{rect. } (a, c) : \text{rect. } (b, c)$ as rect. $(a, c) : \text{rect. } (a, d)$; (v. 3)

$\therefore \text{rect. } (b, c) = \text{rect. } (a, d)$. (v. 4)

Conversely,

If a, b, c, d are four straight lines, such that

$$\text{rect. } (a, d) = \text{rect. } (b, c),$$

$$\text{then } a : b \text{ as } c : d.$$

For the same construction being made,

$$\therefore \text{rect. } (a, d) = \text{rect. } (b, c);$$

$$\therefore \text{rect. } (a, c) : \text{rect. } (b, c) \text{ as rect. } (a, c) : \text{rect. } (a, d); \text{ (v. 2)}$$

$$\text{but rect. } (a, c) : \text{rect. } (b, c) \text{ as } a : b ; \text{ (v. 1)}$$

$$\text{and rect. } (a, c) : \text{rect. } (a, d) \text{ as } c : d ; \text{ (v. 1)}$$

$$\therefore a : b \text{ as } c : d . \text{ (v. 3)}$$

In like manner it may be demonstrated that, *if the rectangle contained by two straight lines is equal to the rectangle contained by two other straight lines, then any proportion having the sides of one rectangle for the extremes and the sides of the other for the means is true.*

$$\text{Thus if } \text{rect. } (a, d) = \text{rect. } (b, c),$$

$$\text{then } b : a \text{ as } d : c,$$

$$\text{and } a : c \text{ as } b : d.$$

COR. If $c = b$, then $\text{rect. } (b, c) = \text{square on } b$.

$$\text{Also when } a : b \text{ as } b : d,$$

the straight lines a, b, d are then said to be proportional.

Hence, *if three straight lines are proportional the rectangle contained by the extremes is equal to the square on the mean: and conversely.*

Hence may be easily established the following Propositions:

PROPOSITION VI.

*If the four straight lines a, b, c, d are proportional,
then $a : c$ as $b : d$.*

For $a : b$ as $c : d$;

$\therefore \text{rect. } (a, d) = \text{rect. } (b, c)$;

$\therefore a : c$ as $b : d$.

PROPOSITION VII.

*If $a : b$ as $c : d$,
then $b : a$ as $d : c$.*

For $\text{rect. } (a, d) = \text{rect. } (b, c)$,

and $\therefore b : a$ as $d : c$.

PROPOSITION VIII.

*If $a : b$ as $c : d$,
then a is $> =$ or $< b$, according as c is $> =$ or $< d$.*

For $\text{rect. } (a, d) = \text{rect. } (b, c)$;

$\therefore$ if c be $> =$ or $< d$, a must be accordingly $> =$ or $< b$.

PROPOSITION IX.

*If $a : b$ as $c : d$,
then a is $> =$ or $< c$, according as b is $> =$ or $< d$.*

This follows in a similar manner to the preceding demonstration.

PROPOSITION X.

If $a : b$ *as* $c : d$,
then $a : b$ *as* $a + c : b + d$.

For rect. $(a, d) = \text{rect. } (b, c)$;
 $\therefore$ rect. $(a, b)(a, d)$ are together = rect. $(b, a)(b, c)$;
 $\therefore$ rect. $(a, b + d) = \text{rect. } (b, a + c)$; (II. 1)
 $\therefore a : b$ *as* $a + c : b + d$.

COR. So $a : b$ *as* the difference of a and c : the difference of b and d .

PROPOSITION XI.

If $a : b$ *as* $c : d$,
and $e : f$,
and also $g : h$,
then $a : b$ *as* $c + e + g : d + f + h$.

$\therefore a : b$ *as* $c : d$;
 $\therefore$ rect. $(a, d) = \text{rect. } (b, c)$;
 and $\therefore a : b$ *as* $e : f$;
 $\therefore$ rect. $(a, f) = \text{rect. } (b, e)$,
 so rect. $(a, h) = \text{rect. } (b, g)$.
 Hence rect. $(a, d)(a, f)(a, h)$ are together
 = rect. $(b, c)(b, e)(b, g)$;
 $\therefore$ rect. $(a, d + f + h) = \text{rect. } (b, c + e + g)$; (II. 1)
 $\therefore a : b$ *as* $c + e + g : d + f + h$.

PROPOSITION XII.

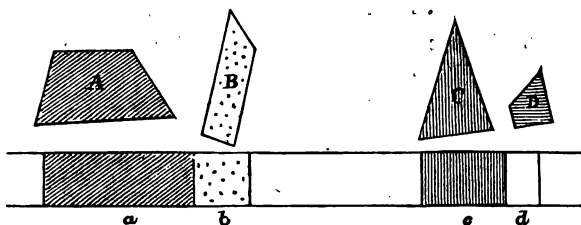
If $a : b$ *as* $c : d$,
and $b : h$ *as* $d : k$,
then $a : h$ *as* $c : k$.

For $a : c$ *as* $b : d$, (v. 6)
 and $b : d$ *as* $h : k$; (v. 6)
 $\therefore a : c$ *as* $h : k$; (v. 3)
 $\therefore a : h$ *as* $c : k$. (v. 6)

Since any lines whatever are equal to certain straight lines, $\therefore$ the above propositions (6—12) will hold for any lines whatever (v. 2).

The preceding Propositions (6—12) may be easily extended to areas, thus :

Let the four rectilinear figures A, B, C, D be proportional, and let a, b, c, d be the bases of rectangles having the same altitude p and areas-respectively $= A, B, C, D$.



Then $\therefore A : B$ as $C : D$;

$\therefore \text{rect. } (p, a) :: \text{rect. } (p, b) \text{ as } \text{rect. } (p, c) : \text{rect. } (p, d)$; (v. 2)

$\therefore a : b$ as $c : d$; (v. 1) and (v. 3)

$\therefore a : c$ as $b : d$; (v. 6)

and $\therefore A : C$ as $B : D$,

so also $b : a$ as $d : c$; (v. 7)

and $\therefore B : A$ as $D : C$.

Again, $A > = \text{or } < B$, according as $a > = \text{or } < b$;

and $\therefore$ as $c > = \text{or } < d$; (v. 8)

and $\therefore$ as $C > = \text{or } < D$.

Similarly, $A > = \text{or } < C$, as $B > = \text{or } < D$. (v. 9)

Again, since $a : b$ as $c : d$;

$\therefore a : b$ as $a + c : b + d$. (v. 10)

Hence $A : B$ as $A + C : C + D$.

So Propositions 11 and 12 may be shewn to hold for rectilineal figures.

Also since the areas of any figures whatever are equal to the areas of certain rectilineal figures; $\therefore$ by (v. 2) the above propositions hold for the areas of any figures whatever.

Also all the preceding theorems may be similarly extended to other magnitudes as solids and angles. For solids may be shewn to be proportional to certain straight lines and angles to certain arcs. Moreover

PROPOSITION XIII.

If A, B, H be areas and c, d, k straight lines, such that

$$\begin{aligned} &A : B \text{ as } c : d, \\ &\text{and } B : H \text{ as } d : k, \\ &\text{then } A : H \text{ as } c : k. \end{aligned}$$

For, if p be any straight line,

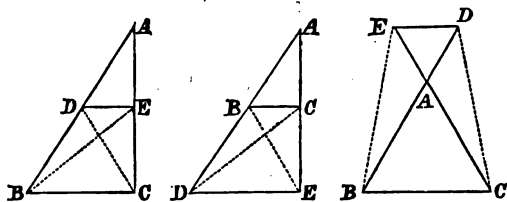
$$\begin{aligned} &\text{then } c : d \text{ as rect. } (p, c) : \text{rect. } (p, d); & (\text{v. 1}) \\ &\therefore A : B \text{ as rect. } (p, c) : \text{rect. } (p, d); & (\text{v. 3}) \\ &\text{so } B : H \text{ as rect. } (p, d) : \text{rect. } (p, k); \\ &\therefore A : H \text{ as rect. } (p, c) : \text{rect. } (p, k); & (\text{v. 12}) \\ &\therefore A : H \text{ as } c : k. & (\text{v. 3}) \end{aligned}$$

BOOK VI.

COMPARISON OF STRAIGHT LINES.

PROPOSITION I.

If a straight line be drawn parallel to one of the sides of a triangle it shall cut the other sides, or these produced, proportionally.



Let DE be drawn $\parallel$ to BC , one of the sides of the $\triangle ABC$.

Then $AD : DB$ as $AE : EC$.

Join BE, DC .

Then $\because DE$ is $\parallel$ to BC ; $\therefore \triangle BDE = \triangle CDE$; (I. 32)

$\therefore \triangle ADE : \triangle BDE$ as $\triangle ADE : \triangle CDE$; (v. 2)

but $\triangle ADE : \triangle BDE$ as $AD : DB$ } (v. 1)

and $\triangle ADE : \triangle CDE$ as $AE : EC$ } (v. 1)

$\therefore AD : DB$ as $AE : EC$. (v. 3)

Conversely,

If the sides AB, AC of the $\triangle ABC$, or these sides produced, be cut proportionally in D, E ,

then shall the straight line joining D, E be $\parallel$ to BC .

The same construction being made,

$$\therefore AD : DB \text{ as } AE : EC,$$

$$\text{and } AD : DB \text{ as } \triangle ADE : \triangle BDE \quad \left. \vphantom{\begin{matrix} AD : DB \\ AE : EC \end{matrix}} \right\} \quad (\text{v. 1})$$

$$\text{also } AE : EC \text{ as } \triangle ADE : \triangle CDE \quad \left. \vphantom{\begin{matrix} AD : DB \\ AE : EC \end{matrix}} \right\} \quad (\text{v. 1})$$

$$\therefore \triangle ADE : \triangle BDE \text{ as } \triangle ADE : \triangle CDE; \quad (\text{v. 3})$$

$$\therefore \triangle BDE \text{ is } = \triangle CDE; \quad (\text{v. 4})$$

$$\therefore DE \text{ is } \parallel \text{ to } BC. \quad (\text{I. 33})$$

COR. Similarly it may be proved that,

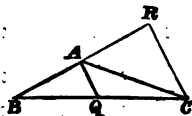
if DE be drawn $\parallel$ to BC ,

then $AD : AB \text{ as } AE : AC$

and conversely.

PROPOSITION II.

The line bisecting the vertical angle of a triangle divides the base into two segments which have the same ratio to one another which the sides of the triangle have.



Let the vertical $\angle BAC$ of the $\triangle ABC$ be bisected by AQ ;

then $BQ : QC$ as $BA : AC$.

Through C draw $CR \parallel$ to QA meeting BA produced in R .

Now $\because QA$ is $\parallel$ to CR ; $\therefore \angle BAQ$ is $= \angle ARC$,
(I. 21)

and $\angle CAQ$ is $= \angle ACR$:
(I. 21)

but $\angle BAQ = \angle CAQ$; $\therefore \angle ARC = \angle ACR$;
 $\therefore AC = AR$. (I. 4)

Also $\because QA$ is $\parallel$ to CR ;

$\therefore BQ : QC$ as $BA : AR$; (VI. 1)

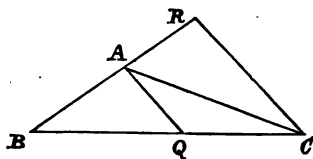
$\therefore BQ : QC$ as $BA : AC$.

Converse Theorem.

If the side BC of $\triangle ABC$ be divided in the point Q so that

$$BQ : QC :: BA : AC,$$

then shall AQ bisect $\angle BAC$.



Through C draw CR $\parallel$ to QA meeting BA produced in R.

Then $\because$ QA is $\parallel$ to CR,

$$\therefore BQ : QC \text{ as } BA : AR, \quad (\text{VI. 1})$$

but $BQ : QC \text{ as } BA : AC;$

$$\therefore BA : AR \text{ as } BA : AC;$$

$$\therefore AR = AC; \quad (\text{v. 4})$$

$$\therefore \angle ACR = \angle ARC. \quad (\text{I. 2})$$

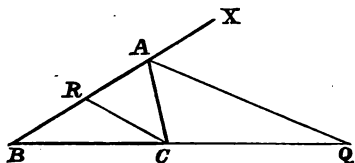
Also $\because$ QA is $\parallel$ to CR;

$$\therefore \angle ACR = \angle CAQ \text{ and } \angle ARC = \angle BAQ; \quad (\text{I. 21})$$

$$\therefore \angle CAQ = \angle BAQ.$$

PROPOSITION III.

If the line bisecting an exterior angle adjacent to the vertical angle of a triangle meet the base produced, then will the segments between the dividing line and the extremities of the base have the same ratio which the sides of the triangle have to one another.



Let AQ bisect the exterior $\angle CAX$ of $\triangle ABC$ and cut BC produced in Q .

Then $BQ : QC$ as $BA : AC$.

Through C draw $CR \parallel$ to QA .

Now $\because CR$ is $\parallel$ to QA , $\therefore \angle XAQ = \angle ARC$, (I. 21)

and $\angle CAQ = \angle ACR$; (I. 21)

but $\angle XAQ = \angle CAQ$, $\therefore \angle ARC = \angle ACR$;

$\therefore AC = AR$. (I. 4)

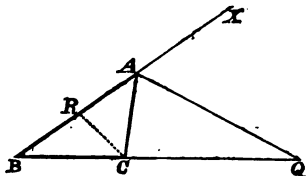
Also $\because CR$ is $\parallel$ to QA ;

$\therefore BQ : QC$ as $BA : AR$; (VI. 1)

$\therefore BQ : QC$ as $BA : AC$.

Converse Theorem.

Let Q be a point, in the side BC of $\triangle ABC$ produced, such that $BQ : QC$ as $BA : AC$. Then shall AQ bisect the exterior $\angle CAX$ adjacent to $\angle BAC$.



Through C draw $CR \parallel$ to QA .

Then $\therefore CR$ is $\parallel$ to QA ,

$$\therefore BQ : QC \text{ as } BA : AR; \quad (\text{vi. 1})$$

$$\text{but } BQ : QC \text{ as } BA : AC;$$

$$\therefore BA : AR \text{ as } BA : AC;$$

$$\therefore AR = AC; \quad (\text{v. 4})$$

$$\therefore \angle ACR = \angle ARC. \quad (\text{I. 2})$$

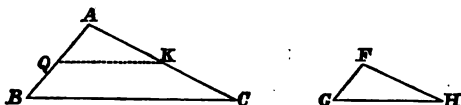
Also $\therefore CR$ is $\parallel$ to QA ,

$$\therefore \angle ACR \text{ is } = \angle CAQ \text{ and } \angle ARC \text{ is } = \angle XAQ; \quad (\text{I. 21})$$

$$\therefore \angle CAQ = \angle XAQ.$$

PROPOSITION IV.

The sides about the equal angles of equiangular triangles are proportional.



Let ABC , FGH be equiangular triangles.

Then shall the sides about the equal $\angle$ s A and F be proportional,
as also the sides about B and G , and those about C and H .

From the arms of the $\angle$ at A , produced if necessary,
cut off $AQ = FG$ and $AK = FH$; join QK .

Then $\triangle s$ AQK , FGH are equal in all respects; (I. 1)

$\therefore \angle AQK = \angle FGH$, and $\therefore = \angle ABC$;

$\therefore BC$ is $\parallel$ to QK ; (I. 20)

$\therefore AB : AQ$ as $AC : AK$; (VI. 1)

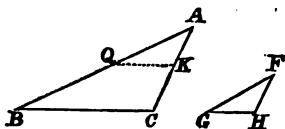
$\therefore AB : AC$ as $AQ : AK$, (V. 6)

and $\therefore$ as $FG : FH$.

Similarly it may be shewn that the sides about B and G
are proportional, as also those about C and H .

Conversely,

If the sides about the angles of one triangle be proportional to the sides about the angles of another, the triangles shall be equiangular.



Let ABC , FGH be two $\triangle s$, such that

$AB : AC$ as $FG : FH$,

and $AB : BC$ as $FG : GH$.

The $\triangle s$ ABC , FGH shall be equiangular.

From the arms of the $\angle$ at A , produced if necessary,
cut off $AQ = FG$, and $AK = FH$; join QK .

Thus $AB : AC$ as $AQ : AK$;

$\therefore AB : AQ$ as $AC : AK$; (v. 6)

$\therefore QK$ is $\parallel$ to BC ; (vi. 1)

$\therefore \triangle s A Q K, A B C$ are equiangular; (i. 21)

$\therefore AQ : QK$ as $AB : BC$,

and $\therefore$ as $FG : GH$;

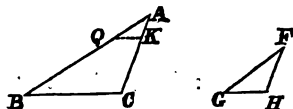
but $AQ = FG$, $\therefore QK = GH$; (v. 4)

$\therefore \triangle FGH$ is equal to $\triangle A Q K$ in all respects; (i. 5)

$\therefore \triangle FGH$ is equiangular to $\triangle ABC$.

Also,

If two $\triangle s$ have one angle of the one equal to one angle of the other, and the sides about the equal angles proportional, then shall the $\triangle s$ be equiangular.



Let ABC, FGH be two $\triangle s$, having $\angle BAC = \angle GFH$,
and such that $AB : AC$ as $FG : FH$.

Then shall $\triangle s ABC, FGH$ be equiangular.

From the arms of the $\angle$ at A , produced if necessary,
cut off $AQ = FG$ and $AK = FH$; join QK .

Then $\triangle s A Q K, FGH$ are equal in all respects,

and $\therefore AB : AC$ as $FG : FH$,

$\therefore AB : AC$ as $AQ : AK$;

$\therefore AB : AQ$ as $AC : AK$; (v. 6)

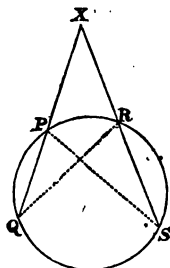
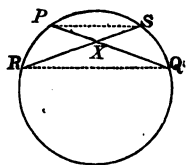
$\therefore QK$ is $\parallel$ to BC ; (vi. 1)

$\therefore \triangle ABC$ is equiangular to the $\triangle A Q K$, (i. 21)

and $\therefore$ to $\triangle FGH$.

PROPOSITION V.

If through any point within or without a circle two secants be drawn, their segments between the given point and the circumference shall be reciprocally proportional.



Through the point X within or without the $\odot PQR$,
let two secants be drawn cutting the $\odot$ in P, Q and
 R, S .

Then $XP : XS$ as $XR : XQ$.

Join PS, RQ :

Then $\therefore$ in $\triangle s PXS, RXQ$,

$\angle PXS = \angle RXQ$, and $\angle PSX = \angle RQX$; (III. 14)

$\therefore \triangle s PXS, RXQ$ are equiangular; (I. 25)

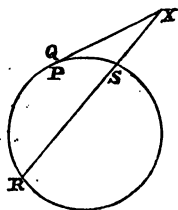
$\therefore XP : XS$ as $XR : XQ$. (VI. 4)

COR. 1. Rectangle $XP, XQ = \text{rectangle } XR, XS$. (v. 5)

COR. 2. Let the point X be without the $\odot$, and let XPQ revolve about X until P and Q coincide;

then the secant XPQ becomes a tangent, and rectangle XP, XQ becomes the square on XQ ;

$\therefore \text{rectangle } XR, XS = \text{square on } XQ$.

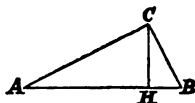


DEFINITION.

Similar rectilineal figures are those which have their corresponding angles equal and the sides about the equal angles proportional.

PROPOSITION VI.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base; the triangles on each side of it shall be similar to the whole triangle and to one another.



Let ACB be a right-angled $\triangle$ having the right $\angle ACB$.

From C let fall $CH \perp$ to AB .

Then shall the $\triangle$ s AHC , CHB , ACB be similar to one another.

For the $\angle A$ is common to the two $\triangle$ s AHC , ACB ;
and right $\angle AHC =$ right $\angle ACB$;

$\therefore \triangle$ s AHC , ACB are equiangular. (I. 25)

Similarly $\triangle$ s CHB , ACB are equiangular.

Hence the $\triangle$ s AHC , CHB , ACB are equiangular and
 $\therefore$ similar to one another. (VI. 4)

COR. *The perpendicular from the right angle is a mean proportional between the segments of the base.*

For since $\triangle AHC$ is equiangular to $\triangle CHB$,

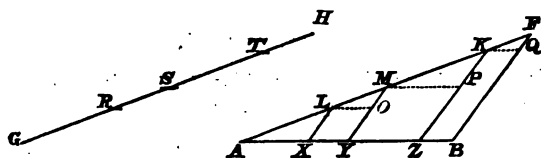
$\therefore AH : HC$ as $HC : HB$; (VI. 4)

$\therefore HC$ is a mean proportional between AH , HB .

C. G.

PROBLEM A.

To divide a straight line similarly to a given divided line.



Let it be required to divide AB similarly to GH .

Through A draw a straight line AF equal to GH , making any $\angle$ with AB ,

and cut off parts $AL, LM, MK = GR, RS, ST$ respectively.

Join BF , and through L, M, K draw $LX, MY, KZ \parallel$ to FB .

Then shall AB be divided similarly to GH .

Through L, M, K draw $LO, MP, KQ \parallel$ to AB .

Then Δ s ALX, LMO, MKP, KFQ are equiangular; (1. 21)

$\therefore AL : AX$ as $LM : LO$ (or XY); (vi. 4)

$\therefore AL : LM$ as $AX : XY$. (v. 6)

In like manner,

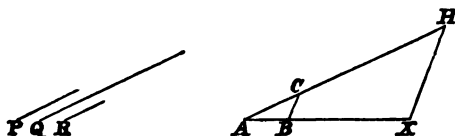
$LM : MK$ as $XY : YZ$,

and $MK : KF$ as $YZ : ZB$.

Hence AB is divided similarly to AF , and $\therefore$ to GH .

PROBLEM B.

To find a fourth proportional to three given straight lines.



Let it be required to find a fourth proportional to the three straight lines P, Q, R .

Draw two st. lines ABX, ACH making any $\angle$ at A .

Cut off $AC = P, CH = Q$, and $AB = R$.

Join CB , and through H draw $HX \parallel$ to CB , cutting ABX in X .

Then shall BX be a fourth proportional to P, Q, R .

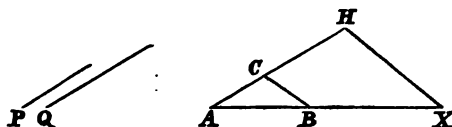
For since HX is $\parallel$ to CB ;

$$\therefore AC : CH \text{ as } AB : BX, \quad (\text{VI. 1})$$

$$\therefore P : Q \text{ as } R : BX.$$

PROBLEM C.

To find a third proportional to two given straight lines.



Let it be required to find a third proportional to the two given straight lines P, Q .

Draw two st. lines ABX, ACH making any $\angle$ at A .

Cut off $AC = P, CH = Q$, and $AB = Q$.

Join CB , and through H draw $HX \parallel$ to CB .

Then shall BX be a third proportional to P, Q .

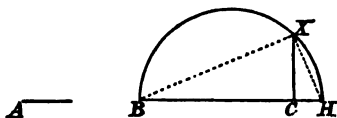
For since HX is $\parallel$ to CB ,

$$\therefore AC : CH \text{ as } AB : BX; \quad (\text{VI. 1})$$

$$\therefore P : Q \text{ as } Q : BX.$$

PROBLEM D.

To find a mean proportional between two given straight lines.



Let it be required to find a mean proportional between A and BC .

Produce BC making the produced part $CH = A$.

On BH describe a semicircle and draw $CX \perp$ to BH .
Then CX shall be a mean proportional between A and BC .

Join BX, XH .

Then $\triangle s BCX, XCH$ are equiangular ;

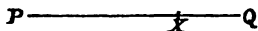
$\therefore CH : CX$ as $CX : BC$; (vi. 4)

$\therefore CX$ is a mean proportional between CH and BC ;

i.e. between A and BC .

PROBLEM E.

Divide a straight line into two parts so that one of the parts shall be a mean proportional between the whole line and the other part.



Let PQ be the given straight line.

Divide it in X so that the rect. PQ, QX may be = square on PX . (ii. B)

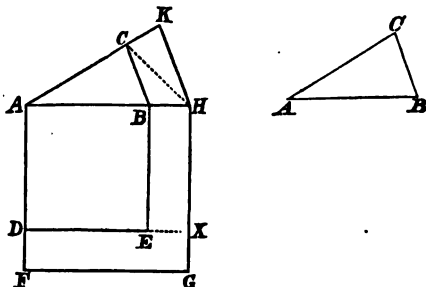
Then $PQ : PX$ as $PX : QX$. (v. 5)

PQ is said to be divided in extreme and mean ratio in X .

COMPARISON OF AREAS OF SIMILAR RECTILINEAL FIGURES.

PROPOSITION VII.

Similar triangles are to one another as the squares on their corresponding sides.



Let $\triangle ABC$, $\triangle AHK$ be similar $\triangle s$;

Then $\triangle ABC : \triangle AHK$ as square on $AB : \text{square on } AH$ the corresponding side of $\triangle AHK$.

Let the $\triangle ABC$ be applied to the $\triangle AHK$ so as to have the sides AB , AC on the corresponding sides AH , AK ; and $\therefore$ having the side $BC \parallel$ to HK .

On AB , AH describe the squares AE , AG .

Produce DE to meet GH in X ; and join CH .

Then $\triangle ABC : \triangle AHC$ as $AB : AH$, and $\therefore$ as $AE : AX$;
(v. 1)

also $\triangle AHC : \triangle AHK$ as $AC : AK$;
(v. 1)

and $\therefore$ as $AB : AH$;
(vi. 1)

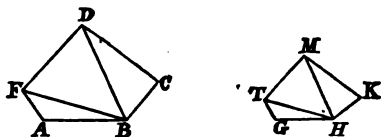
i.e. as $AD : AF$;

and $\therefore$ as $AX : AG$;
(v. 1)

$\therefore \triangle ABC : \triangle AHK$ as $AE : AG$.
(v. 12)

PROPOSITION VIII.

Similar polygons are to one another as the squares on their corresponding sides.



Let $ABCDF$, $GHKMT$ be similar polygons.

Then shall they bear to one another the same ratio as the squares on their corresponding sides AB , GH .

Divide the polygons into triangles by straight lines drawn from B and H .

Then $\therefore FA : AB$ as $TG : GH$, and $\angle FAB = \angle TGH$;

$\therefore \Delta s$ FAB , TGH are similar; (vi. 4)

and are to each other as squares on FB , TH . (vi. 7)

Again, $\therefore \angle DFA = \angle MTG$ and $\angle BFA = \angle HTG$;

$\therefore \angle DFB = \angle MTH$;

also $DF : FA$ as $MT : TG$;

and $FA : FB$ as $TG : TH$;

$\therefore DF : FB$ as $MT : TH$; (v. 12)

$\therefore \Delta s$ DFB , MTH are similar; (vi. 4)

and are to each other as squares on FB , TH , (vi. 7)

and are $\therefore$ in the same ratio as Δs FAB , TGH .

So each of the Δs in the polygon $ABCDF$: the corresponding Δ in the polygon $GHKMT$ in the same ratio.

$\therefore$ all the Δs in the polygon $ABCDF$ bear to all the Δs in polygon $GHKMT$ the same ratio. (v. 11)

That is, the area of polygon $ABCDF$: area of polygon $GHKMT$ as ΔFAB : ΔTGH ,

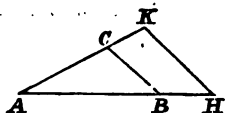
and $\therefore$ as square on AB : square on GH . (vi. 7)

COR. Each of the sides of the one polygon is to the corresponding side of the other as $AB : GH$.

$\therefore$ also the perimeter of the one is to the perimeter of the other in the same ratio. (v. 11)

PROPOSITION IX.

If four straight lines are proportional, the squares on these lines are proportional.



Let AB, AH, AC, AK be cut off from the arms of any $\angle A =$ the given st. lines and $\therefore$ proportional.

Then shall the squares on AB, AH, AC, AK be proportional also.

Join BC, HK .

Then BC is $\parallel$ to HK , (vi. 1)

and $\therefore \triangle s ABC, AHK$ are similar; (I. 21)

$\therefore \triangle ABC : \triangle AHK$ as square on AB : square on AH ,
(vi. 7)

and also as square on AC : square on AK ;
(vi. 7)

$\therefore$ sq. on AB : sq. on AH as sq. on AC : sq. on AK . (v. 3)

COR. Hence also, if similar figures are described on AB and AH and similar figures on AC and AK ,

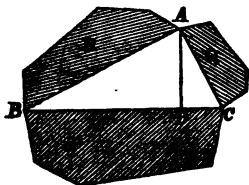
since figure on AB : similar figure on AH as square on AB : square on AH ,

and figure on AC : similar figure on AK as square on AC : square on AK , it follows that

fig. on AB : sim. fig. on AH as fig. on AC : sim. fig. on AK .
(v. 3)

PROPOSITION X.

In right-angled triangles the rectilineal figure described on the hypotenuse is equal to the similar and similarly described figures on the other two sides.



Let P, Q, R be figures similar and similarly described on the sides of the right-angled $\triangle ABC$.

Then the figure P on the hypotenuse BC is equal to the figures Q and R together.

From A let fall $AD \perp$ to BC .

Then on BC, AC are the similar $\triangle$ s ABC, DAC ,

and also the similar figures P, Q ;

$$\therefore \triangle ABC : \triangle DAC \text{ as } P : Q; \text{ (vi. 9 Cor.)}$$

$$\begin{aligned} \therefore \triangle ABC : P \text{ as } \triangle DAC : Q, \\ \text{so also } \triangle ABC : P \text{ as } \triangle DBA : R; \end{aligned} \quad \text{(v. 6)}$$

$$\therefore \triangle ABC : P \text{ as } \triangle DAC \text{ together with } \triangle DBA : \text{figures } Q, R \text{ together,} \quad \text{(v. 11)}$$

but $\triangle ABC$ is $= \triangle$ s DAC, DBA together;

$$\therefore P \text{ is } = Q \text{ and } R \text{ together.} \quad \text{(v. 4)}$$

PROBLEM F.

On a given straight line describe a figure similar and similarly situated to a given rectilineal figure.



Let $CDEFG$ be the given rectilineal figure,
and XY the given straight line.

It is required to describe on XY a rectilineal figure similar and similarly situated to $CDEFG$.

Join DG , DF ,

and make $\angle s YXR, XYR = \angle s DCG, CDG$, respectively,

$\angle s YRK, RYK = \angle s DGF, GDF$,

and $\angle s YKH, KYH = \angle s DFE, FDE$.

Then shall $XYHKR$ be similar and similarly situated to $CDEFG$.

For the $\Delta s GCD, GDF$ &c. are respectively equiangular to the $\Delta s RXY, RYK$ &c.

$\therefore CG : GD$ as $XR : RY$, (vi. 4)

and $GD : GF$ as $RY : RK$; (vi. 4)

$\therefore CG : GF$ as $XR : RK$, (v. 12)

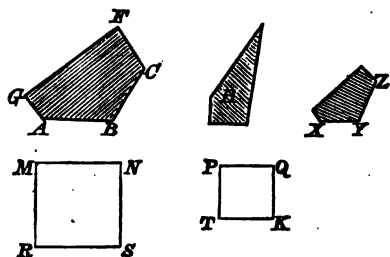
and so on.

Thus all the $\angle s$ in the one polygon are = the corresponding $\angle s$ in the other, and the sides about the corresponding $\angle s$ are proportional.

$\therefore$ the polygons are similar.

PROBLEM G.

To describe a rectilineal figure equal to one and similar to another given rectilineal figure.



Let it be required to describe a rectilineal figure similar to $ABCFG$ and equal to H .

Describe squares MS , PK equal to AC , and H . (II. A)

To MN , PQ and AB take a fourth proportional XY . (VI. B)

On XY describe a figure XZ similar to that on AB : (VI. F)

this shall be the figure required.

For since $MN : PQ$ as $AB : XY$;

$\therefore$ square MS : square PK as figure AC : figure XZ ;
(VI. 9 Cor.)

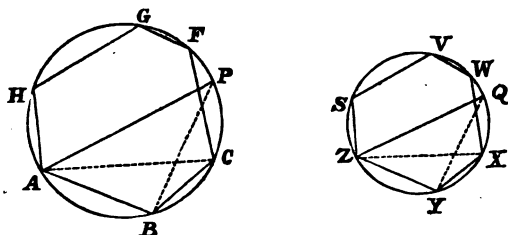
but square MS = figure AC , $\therefore$ square PK = figure XZ ;
(v. 4)

$\therefore$ figure XZ = figure H , and it is similar to figure AC .

COMPARISON OF CIRCLES.

PROPOSITION XI.

Similar polygons inscribed in circles are to one another as the squares on their diameters, also the perimeters of the polygons are as the diameters of the circles.



Let ACG , ZXV be similar polygons inscribed in the $\odot$ s AG , ZV .

Through A , Z draw the diameters AP , ZQ .

Then polygon ACG : polygon ZXV as square on AP : square on ZQ .

Join BP , YQ , also AC , ZX .

Then $\angle ACB = \angle ZXY$; (VI. 8)

$\therefore \angle APB = \angle ZQY$, (III. 14)

also $\angle PBA = \angle QYZ$; (III. 16)

$\therefore \triangle APB$ is equiangular to $\triangle ZQY$; (I. 25)

$\therefore AP : ZQ$ as $AB : ZY$; (VI. 4)

$\therefore$ square on AP : square on ZQ as square on AB : square on ZY . (VI. 9)

And since polygon ACG : polygon ZXV as square on AB : square on ZY ; (VI. 8)

$\therefore$ polygon ACG : polygon ZXV as square on AP : square on ZQ . (v. 3)

Also the perimeters of the polygons are as the diameters,

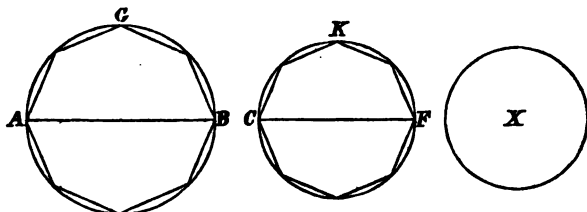
for $AB : ZY$ as $AP : ZQ$; (VI. 4)

also $BC : YX$ in same ratio, and so on;

$\therefore$ perimeter of ACG : perimeter of ZXV in same ratio. (v. 11)

PROPOSITION XII.

Circles are to one another as the squares on their diameters, and their circumferences are as their diameters.



Let ABG , CFK be two circles; AB , CF their diameters.

Then shall $\odot ABG$, CFK be to one another as the squares on AB , CF .

For if not, if possible, suppose that

square on AB : square on CF as $\odot ABG$: some space X less than $\odot CFK$.

Inscribe in the $\odot CFK$ a polygon which differs from the $\odot CFK$ by a space less than the excess of $\odot CFK$ above the space X ;

(iv. 14, Cor. 2)

so that the polygon CFK is $>$ the space X .

Now let a similar polygon to this be inscribed in $\odot ABG$.

(iv. 10)

Then polygon in $\odot ABG$: polygon in $\odot CFK$ as square on AB : square on CF ,

(vi. 11)

and $\therefore$ as $\odot ABG$: X ;

(hyp.)

but polygon in $\odot ABG$ is $<$ $\odot ABG$, $\therefore$ polygon in $\odot CFK$ is $<$ X ;

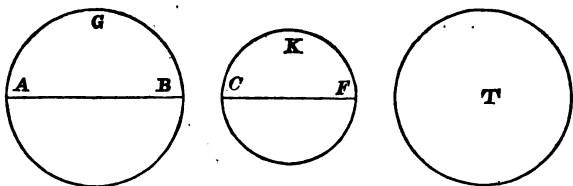
(v. 9)

but it is also greater, which is impossible;

$\therefore$ square on AB is not to square on CF as $\odot ABG$: any space less than $\odot CFK$.

Again, if possible, suppose that

square on AB : square on CF as $\odot ABG$: some space T , greater than $\odot CFK$;



$\therefore$ square on CF : square on AB as T : $\odot ABG$, (v. 7)
and $\therefore$ as $\odot CFK$: some space less than $\odot ABG$; (v. 9)
which is impossible.

Hence the ratio of the square on AB to the square on CF is not the same as that of the $\odot ABG$ to any space less than $\odot CFK$, nor as that of the $\odot ABG$ to any space greater than $\odot CFK$,

and is $\therefore$ the same as that of the $\odot ABG$ to the $\odot CFK$.

Also,

The circumferences of circles are to one another as their diameters.

For the perimeters of similar polygons inscribed in circles are as their diameters. (vi. 11)

Also a polygon may be inscribed in a $\odot$ whose perimeter shall differ from the $\odot$ ce of the $\odot$ by less than any assignable magnitude. (iv. 15, Cor. 2)

Hence it may be shewn by a demonstration precisely similar to the foregoing, that the circumferences of circles are as their diameters.

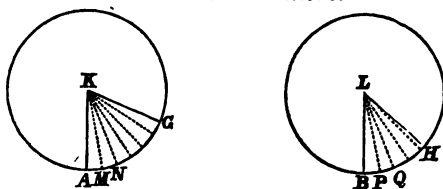
COMPARISON OF ANGLES.

DEFINITION.

A sector is the figure contained by an arc of a circle and the radii drawn to its extremities.

PROPOSITION XIII.

*In equal circles, angles at the centre have the same ratio as the arcs on which they stand;
so also have the sectors.*



Let AKG , BLH be angles at the centres K , L of equal circles.

Then shall $\angle AKG$ have to $\angle BLH$ the same ratio as arc AG to arc BH .

For if $\angle AKG$ be divided into any number whatever of equal $\angle$ s AKM , MKN , &c., and $\angle$ s BLP , PLQ , &c. equal to them be successively cut off from $\angle BLH$:

then the arc AG will contain the same number of arcs each equal to AM as $\angle AKG$ does angles each equal to $\angle AKM$;

also the arc BH will contain the same integral number of arcs each equal to AM as $\angle BLH$ does angles each equal to $\angle AKM$;

$\therefore \angle AKG : \angle BLH$ as arc $AG : \text{arc } BH$. (Def.)

So also sector $AKG : \text{sector } BLH$ as arc $AG : \text{arc } BH$.

For by applying the sector KAM to the sectors KMN , LBP , &c. it may be shewn that they are equal to one another.

The remainder of the proof is similar to the foregoing.

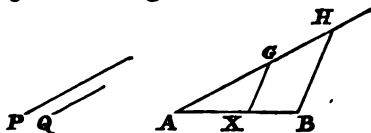
SUPPLEMENT TO BOOK VI.

PROBLEM (a).

To divide a straight line in a given ratio.

Let AB be a given straight line.

It is required to divide AB in the ratio of P to Q .



Through A draw a straight line AGH , making any $\angle$ with AB ;

cut off $AG = P$ and $GH = Q$.

Join HB , and through G draw $GX \parallel$ to HB .

Then shall AB be divided in X in the ratio of P to Q .

$\therefore GX$ is $\parallel$ to HB ; $\therefore AX : XB$ as $AG : GH$; (VI. 1)
i. e. as $P : Q$.

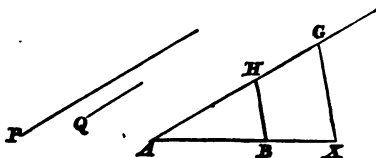
Hence AB has been divided in the required ratio.

PROBLEM (b).

To produce a given straight line to a point such that the whole line thus produced shall be to the part produced in a given ratio.

Let AB be the given straight line.

It is required to produce AB to a point such that the whole line thus produced shall be to the part produced in the ratio of P to Q .



Through A draw any straight line AHG making an $\angle$ with AB ;

cut off $AG = P$ and $GH = Q$.

Join HB , and through G draw $GX \parallel$ to HB meeting AB produced in X .

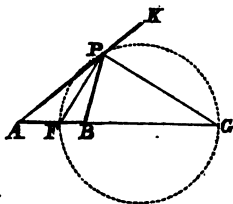
Then $AX : XB$ as $P : Q$.

$\therefore GX$ is $\parallel$ to HB ;
 $\therefore AX : XB$ as $AG : GH$; (VI. 1)
 $\therefore AX : XB$ as $P : Q$.

PROBLEM (c).

Find the locus of a point whose distances from two fixed points are always in a constant ratio.

Let A, B be the two fixed points.
Join AB , and divide AB in F
in the given ratio;
also produce AB to G so that
 $AG : GB$ in the given ratio.



Now let P be any point in the locus; join AP, PB, PF, PG ; and produce AP to K .

Then $AP : PB$ as $AF : FB$;

$\therefore PF$ bisects the $\angle APB$; (vi. 2)

also $AP : PB$ as $AG : GB$;

$\therefore PG$ bisects the $\angle BPK$. (vi. 3)

Hence the $\angle FPG$ is a right $\angle$. (I. 9)

$\therefore$ every point in the locus is situated in the $\bigcirc$ ce of the $\odot$ of which FG is the diameter. (III. 16). (III. f)

Also $AF : FB$ as $AG : GB$;

$\therefore AF : AG$ as $FB : GB$;

$\therefore FG$ is divided in B in the ratio of AF to AG .

Again, every point in the $\bigcirc$ ce is a point in the locus.

For let P be any point in the $\bigcirc$ ce. Join AP, PF .

Then the straight line drawn from P making an $\angle$ with $PF = \angle APF$ will divide FG in the ratio of AF to AG , and $\therefore$ in the ratio of FB to GB , and $\therefore$ cuts FG in B ;

i.e. $\angle BPF = \angle APF$;

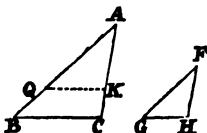
$\therefore AP : PB$ as $AF : FB$; (vi. 2)

$\therefore$ every point in the $\bigcirc$ ce is a point in the locus;

the locus is the $\bigcirc$ ce of the $\odot$ of which FG is the diameter.

THEOREM (d).

If two triangles have one angle of the one equal to one angle of the other and the sides about two other angles proportional, then shall the triangles be equiangular in certain cases.



Let ABC, FGH be two triangles having the $\angle ABC = \angle FGH$ and the sides about the $\angle$ s BAC, GFH proportional,

so that $BA : AC$ as $GF : FH$;

then shall the Δ s ABC, FGH be equiangular in certain cases.

Cut off $AQ = FG$ and $AK = FH$; and join QK .

Then $BA : AC$ as $QA : AK$; (v. 2)

$\therefore BA : QA$ as $AC : AK$; (v. 6)

$\therefore QK$ is $\parallel$ to BC ; (vi. 1)

$\therefore \angle AQB$ is $= \angle ABC$, and $\angle AKQ = \angle ACB$; (i. 21)

$\therefore \Delta AQB$ is equiangular to ΔABC ;

and since $\angle ABC = \angle FGH$, $\therefore \angle AQB = \angle FGH$;

also $AQ, AK = FG, FH$ respectively;

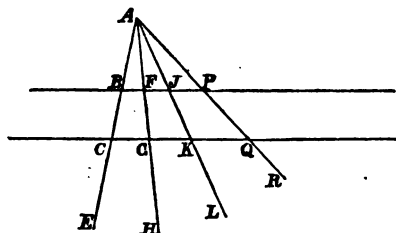
$\therefore \Delta$ s AQB, FGH are equal in all respects if the $\angle$ s AQB, FHG are both acute, both obtuse, or if one of them is a right $\angle$; (i. b)

$\therefore$ if the $\angle$ s ACB, FHG are both acute, both obtuse, or if one of them is a right $\angle$, then ΔABC is equiangular to ΔFGH .

C. G.

THEOREM (c).

If any number of straight lines be drawn from a given point and be cut by parallel straight lines, then shall these parallel straight lines be divided proportionally.



Let the straight lines AE, AH, AL, AR be cut by the parallel straight lines BP, CQ .

Then shall BP, CQ be divided proportionally.

For $BF : CG$ as $AF : AG$, (vi. 4)

and $\therefore$ as $FJ : GK$; (vi. 4)

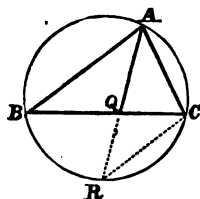
$\therefore BF : FJ$ as $CG : GK$. (v. 6)

Similarly $FJ : JP$ as $GK : KQ$.

Hence BP, CQ are divided proportionally.

THEOREM (f).

If an angle of a triangle be bisected by a straight line which likewise cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base together with the square on the straight line which bisects the angle.



Let the $\angle BAC$ of $\triangle ABC$ be bisected by AQ .

Then the rectangle $(BA, AC) = \text{rectangle } (BQ, QC)$
 + square on AQ .

About ABC describe a $\odot$;

produce AQ to meet the $\odot$ in R and join CR .

Then $\angle s \ ABC, \ ARC$ in the same segment are equal,
 (III. 14)

also $\angle BAQ = \angle RAC$;

$\therefore \triangle s \ BAQ, \ RAC$ are equiangular; (I. 25)

$\therefore BA : AQ \text{ as } AR : AC$; (VI. 4)

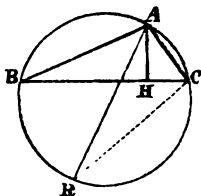
$\therefore \text{rectangle } (BA, AC) = \text{rectangle } (AQ, AR)$ (v. 5)

$= \text{rect. } (AQ, QR) + \text{square on } AQ,$
 (II. 1)

and $\therefore$ $= \text{rect. } (BQ, QC) + \text{square on } AQ.$
 (VI. 5)

THEOREM (8).

If from the vertex of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle about the triangle.



Let ABC be the given Δ , $ABRC$ the $\odot$ described about it, AR the diameter through A , and $AH \perp$ to BC .

Then rectangle $(BA, AC) = \text{rectangle } (AH, AR)$.

Join RC .

Then $\angle s \ ABC, \ ARC$ in the same segment are equal ;
(III. 14)

also the right $\angle s \ AHB, \ ACR$ are equal ; (III. 16)

$\therefore \Delta s \ AHB, \ ACR$ are equiangular ; (I. 25)

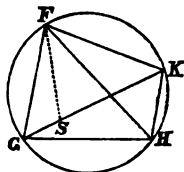
$\therefore BA : AH \text{ as } AR : AC ;$ (VI. 4)

rectangle $(BA, AC) = \text{rectangle } (AH, AR)$.

(v. 5)

THEOREM (h).

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to both the rectangles contained by the opposite sides.



Let $FGHK$ be a quadrilateral figure inscribed in the circle $\odot FGHK$, and FH, GK its diagonals.

Then rectangle (FH, GK) = rectangle (FG, HK) + rectangle (GH, FK) .

Make $\angle GFS = \angle HPK$.

Then since $\angle s$ FGK, FHK in the same segment are equal ;

(III. 14)

$\therefore \Delta s$ FGS, FHK are equiangular ;

(I. 25)

$\therefore FG : GS$ as $FH : HK$;

(VI. 4)

$\therefore$ rectangle (FG, HK) = rectangle (FH, GS) .

(V. 5)

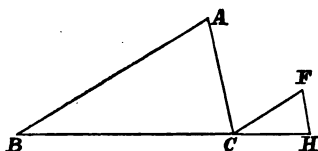
Similarly rectangle (GH, FK) = rectangle (FH, SK) ;

$\therefore$ rectangle (FG, HK) + rectangle (GH, FK) = rectangle (FH, GS) + rectangle (FH, SK) = rectangle (FH, GK) .

(II. 1)

THEOREM (i).

If two triangles which have two sides of the one proportional to two sides of the other be joined at one angle, so as to have the former sides parallel to the latter respectively: the remaining sides shall be in a straight line.



Let ABC , FCH be two Δ s, having the sides about the $\angle$ s A and F proportional, so that $BA : AC$ as $CF : FH$, be joined at one angle C , and have the sides BA , AC parallel to the sides CF , FH respectively.

Then shall BC , CH be in one straight line.

For $\angle BAC = \angle ACF$ and $\therefore = \angle CFH$, (I. 21)
and the sides about the $\angle$ s at A and F are proportional;

$\therefore$ the Δ s ABC , FCH are equiangular; (VI. 4)

$\therefore \angle ACB = \angle FHC$,

also $\angle ACF = \angle CFH$;

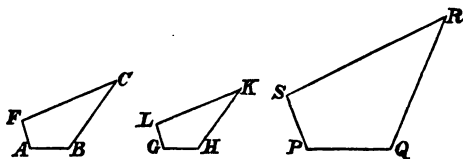
$\therefore$ the $\angle$ s at C are = the three $\angle$ s of ΔFCH ,

and are $\therefore =$ two right $\angle$ s; (I. 24)

$\therefore BC$, CH are in one straight line. (I. 10)

THEOREM (j).

Rectilineal figures which are similar to the same rectilineal figure are similar to one another.



Let ABC, GHK be each similar to PQR ,
then shall ABC, GHK be similar to one another.

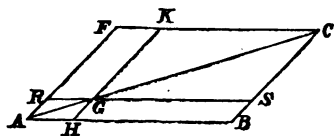
For ABC, GHK are each equiangular to PQR and
 $\therefore$ to one another.

Also the sides about the $\angle$ s of ABC are proportional
to the sides about the corresponding $\angle$ s of PQR , and are
therefore proportional to the sides about the corresponding
 $\angle$ s of GHK . (v. 3)

Hence ABC is similar to GHK .

THEOREM (k).

Parallelograms about the diameter of any parallelogram are similar to the whole and to one another.



Let RH, KS be $\square$ s about the diameter AC of $\square FB$,
 RH, KS shall be similar to FB and to one another.

For $\triangle$ s AHG, ABC are equiangular;

$$\therefore AH : HG \text{ as } AB : BC. \quad (\text{vi. 4})$$

Hence the $\square$ s RH, FB are equiangular and have the sides about their equal angles proportional;

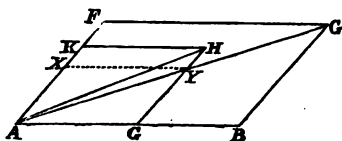
$$\therefore RH \text{ is similar to } FB.$$

So KS is similar to FB ;

$$\therefore RH \text{ is similar to } KS. \quad (\text{vi. } j)$$

THEOREM (I).

If two similar parallelograms have a common angle and be similarly situated, they are about the same diameter.



Let KG , FB be similar $\square$ s having a common $\angle$ at A and similarly situated.

Then shall they be about the same diameter through A .

For if not, if possible, let the diameters AH , AC not coincide; let HG cut AC in Y ; through Y draw $YX \parallel$ to AB .

Then $\square XG$ is similar to $\square FB$; (vi. k)

but KG is similar to FB ; (hyp.)

$\therefore XG$ is similar to KG ; (vi. j)

$\therefore AG : AX$ as $AG : AK$;

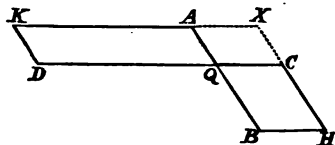
$\therefore AX = AK$; (v. 4)

which is impossible.

Hence KG , FB cannot but be about the same diameter.

THEOREM (m).

Parallelograms which are equal and equiangular have the sides about their equal angles reciprocally proportional.



Let QH, QK be equal and equiangular parallelograms having the angles BQC, AQD equal to one another.

Then $QA : QB$ as $QC : QD$.

Let the $\square$ s be placed so that AQ, QB may be in the same straight line, and also DQ, QC in the same straight line;

produce KA, HC to meet in X : then QX is a $\square$.

Now $\therefore \square QH = \square QK$;

$\therefore \square QX : \square QH$ as $\square QX : \square QK$; (v. 2)

but $\square QX : \square QH$ as $QA : QB$;

and $\square QX : \square QK$ as $QC : QD$;

$\therefore QA : QB$ as $QC : QD$. (v. 3)

Conversely, If the sides about the equal angles BQC, AQD of the equiangular parallelograms QH, QK are reciprocally proportional,

then $\square QH = \square QK$.

Now $QA : QB$ as $QC : QD$.

But $QA : QB$ as $\square QX : \square QH$;

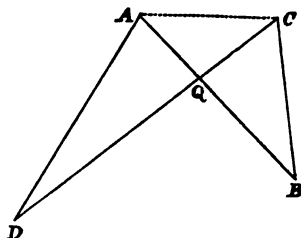
and $QC : QD$ as $\square QX : \square QK$;

$\therefore \square QX : \square QH$ as $\square QX : \square QK$; (v. 3)

$\therefore \square QH = \square QK$. (v. 4)

THEOREM (n).

Equal triangles having one angle of the one equal to one angle of the other, have the sides about those angles reciprocally proportional.



Let AQD, BQC be equal triangles, having the $\angle AQD = \angle BQC$.

Then $QA : QB$ as $QC : QD$.

Let the Δ s be so placed that QA may be in the same straight line with QB and QC with QD .

Join AC .

Then $\because \Delta AQD = \Delta BQC$;

$\therefore \Delta AQC : \Delta BQC$ as $\Delta AQC : \Delta AQD$;

but $\Delta AQC : \Delta BQC$ as $QA : QB$,
and $\Delta AQC : \Delta AQD$ as $QC : QD$;

(v. 1)

$\therefore QA : QB$ as $QC : QD$. (v. 3)

Conversely, If AQD, BQC be two Δ s having the $\angle$ s $AQD, BQC =$ one another, and the sides about those angles reciprocally proportional,

then $\Delta AQD = \Delta BQC$.

For $QA : QB$ as $QC : QD$.

But $QA : QB$ as $\Delta AQC : \Delta BQC$,
and $QC : QD$ as $\Delta AQC : \Delta AQD$;

(v. 1)

$\therefore \Delta AQC : \Delta BQC$ as $\Delta AQC : \Delta AQD$;

(v. 3)

$\therefore \Delta BQC = \Delta AQD$. (v. 4)

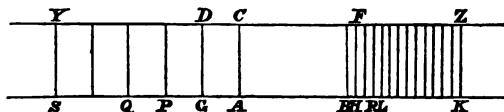
Euclid's test of proportion is the following :

The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatever of the first and third being taken and any equimultiples whatever of the second and fourth, if the multiple of the first be equal to that of the second, then the multiple of the third is equal to that of the fourth, if greater, greater, and if less, less.

Prop. I. Book V., and Prop. XIII. Book VI., may be established by the direct application of the above test.

THEOREM (o). PROP. I. BOOK V.

Straight lines have to one another the same ratio as the rectangles of equal altitude described upon them.



Let AD , BF be rectangles of equal altitude on the straight lines AG , BH .

Then shall AG have to BH the same ratio as AD to BF .

From AG produced cut off any number of parts GP , PQ , &c. each = AG , and from BH produced any number of parts HR , RL , &c. each = BH .

Complete the rectangles PD , FR , &c.

Thus any equimultiples whatever have been taken of AG and rectangle AD ; namely, AS and rectangle AY ;

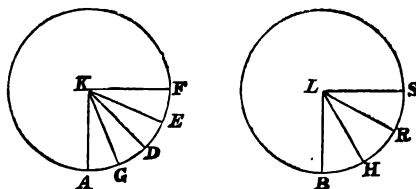
and any equimultiples whatever have been taken of BH and rectangle BF ; namely, BK and rectangle BZ .

Also if AS be = BK , then rectangle AY is = rectangle BZ , if greater, greater, and if less, less;

$\therefore AG : BH$ as $AD : BF$.

THEOREM (*p*). PROP. XIII. BOOK VI.

In equal circles, angles at the centre have the same ratio as the arcs on which they stand; so also have the sectors.



Let $\angle AKG$, $\angle BLH$ be angles at the centres K , L of equal circles.

Then shall $\angle AKG$ have the same ratio to $\angle BLH$ as arc AG to arc BH .

For if $\angle$ s GKD , DKE , &c. be made each equal to $\angle AKG$, and $\angle$ s HLR , &c. each equal to $\angle BLH$;

then $\angle AKF$ and arc AF are equimultiples of $\angle AKG$ and arc AG , and $\angle BLS$ and arc BS are equimultiples of $\angle BLH$ and arc BH .

Thus any equimultiples have been taken of $\angle AKG$ and arc AG , and any equimultiples of $\angle BLH$ and arc BH ;

also if $\angle AKF$ be $= \angle BLS$, then arc AF is $=$ arc BS , if greater, greater, and if less, less.

$\therefore \angle AKG : \angle BLH$ as arc $AG : \text{arc } BH$.

Similarly it may be shewn that

sector $AKG : \text{sector } BLH$ as arc $AG : \text{arc } BH$.

COMPOUND RATIOS.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them **the ratio compounded of the ratios** of the 1st to the 2nd, the 2nd to the 3rd, and so on to the last (or of ratios the same as these ratios).

Thus, if there are three magnitudes of the same kind, the 1st has to the 3rd the ratio compounded of the ratios of the 1st to the 2nd, and the 2nd to the 3rd.

Moreover, if these three magnitudes are proportional, then the ratio of the 1st to the 2nd is the same as the ratio of the 2nd to the 3rd.

Accordingly, when three magnitudes are proportional, the 1st is said to have to the 3rd

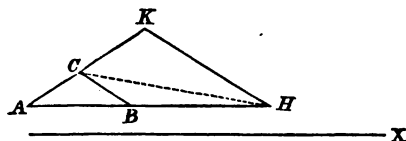
the **duplicate ratio** of that which it has to the 2nd,

The antecedent terms of proportionals are said to be **homologous** to one another; so also are the consequents.

Thus, the corresponding sides of similar figures are homologous to one another.

PROPOSITION (q).

Similar triangles are to one another in the duplicate ratio of their homologous sides.



Let the similar triangles ABC , AHK be placed so as to have the sides AB , AC along the homologous sides AH , AK and $\therefore BC \parallel$ to HK .

Join CH , and take X a third proportional to AB and AH .

Then $\triangle ABC : \triangle AHC$ as $AB : AH$, (v. 1)

and $\triangle AHC : \triangle AHK$ as $AC : AK$, (v. 1)

and $\therefore$ as $AB : AH$, (vi. 1)

and $\therefore$ as $AH : X$; (v. 3)

$\therefore \triangle ABC : \triangle AHK$ as $AB : X$; (v. 13)

i.e. $\triangle ABC : \triangle AHK$ in the duplicate ratio of AB to AH .

Hence, proceeding as in Proposition VIII., it may be proved that

Similar polygons are to one another in the duplicate ratio of their homologous sides.

Thus squares are to one another in the duplicate ratio of their sides.

PROPOSITION (r).

If four straight lines are proportional, the duplicate ratio of the first two is the same as that of the latter two.

$$\begin{array}{l} a \\ b \\ x \end{array} \quad \begin{array}{l} c \\ d \\ z \end{array}$$

Let a, b, c, d be the four given straight lines.

To a and b take a 3rd proportional x , and to c and d a 3rd proportional z .

Then $\therefore a : b$ as $c : d$;

$$\therefore b : x \text{ as } d : z; \quad (\text{V. 3})$$

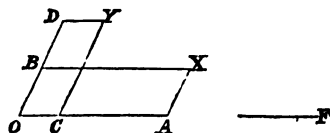
$$\therefore a : x \text{ as } c : z, \quad (\text{V. 12})$$

i.e. the duplicate ratio of a to b is the same as that of c to d .

COR. *If four straight lines are proportional, similar rectilineal figures on the first two are proportional to similar rectilineal figures on the last two.* (VI. q)

PROPOSITION (s).

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.



Let OX , OY be equiangular parallelograms, placed so as to have the arms of their equal angles AOB , COD in the same straight lines.

Then $\square OX$ shall have to $\square OY$ the ratio compounded of the ratios of

OA to OC and OB to OD .

Take F a fourth proportional to OB , OD and OC .

Then $\square OX : \square BC$ as $OA : OC$, (v. 1)

and $\square BC : \square OY$ as $OB : OD$, (v. 1)

and $\therefore$ as $OC : F$; (v. 3)

$\therefore \square OX : \square OY$ as $OA : F$, (v. 13)

which is the ratio compounded of the ratios of

OA to OC and OB to OD .

SOLID GEOMETRY.

PLANES.

DEFINITION.

A solid has length, breadth, and thickness.

A solid is bounded by superficies.

PROPOSITION I.

One part of a straight line cannot be in a plane, and another part out of it.



For if it be possible,

let one part AB of the straight line ABC be in the plane PN , and another part BC out of it.

In the plane PN , produce AB in a straight line to X .

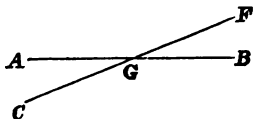
Let a plane through ABX be turned about ABX till it passes through C , and $\therefore$ contains BC .

Then the two straight lines ABC , ABX in one plane have a common segment ; which is impossible.

$\therefore$ *One part of a straight line, &c.*

PROPOSITION II.

Two straight lines which cut one another are in one plane.



Let the two straight lines AB , CF cut one another in G .

Then shall AB , CF lie in one plane.

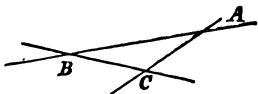
For let a plane through AB be turned about AB till it passes through C .

Then since this plane contains C and G , it contains the straight line CGF .

Hence AB , CF lie in this plane.

PROPOSITION III.

Three straight lines which cut one another and form a triangle are in one plane.



Let the three straight lines AB , BC , CA cut one another and form a $\triangle$.

Then shall AB , BC , CA lie in one plane.

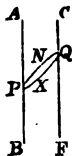
For let a plane containing BC be turned about BC till it passes through A .

Then since this plane contains A and B it contains AB , and since it contains C and A it contains CA .

Hence AB , BC , CA lie in this plane.

PROPOSITION IV.

If two straight lines are parallel, the straight line drawn from any point in the one to any point in the other is in the same plane with the parallels.



Let AB , CF be parallel straight lines, P , Q any points in AB , CF respectively.

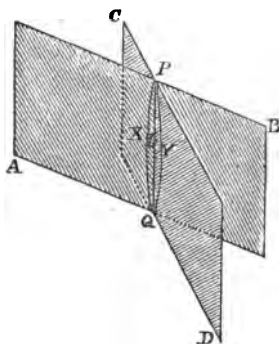
Then shall the straight line joining P , Q be in the same plane as AB , CF .

For if not, if possible, let it be out of this plane as PXQ , and in this plane draw a straight line PNQ joining P and Q , then there are two straight lines PNQ , PXQ inclosing a space; which is impossible.

Hence, *If two straight lines, &c.*

PROPOSITION V.

If two planes cut one another, their common section is a straight line.



Let PIQ be the common section of two planes AB , CD which cut one another; PIQ shall be a straight line.

For if not, if possible,

let a straight line PXQ be drawn from P to Q in the plane AB , and a straight line PYQ from P to Q in the plane CD .

Then there are two straight lines PXQ , PYQ inclosing a space; which is impossible.

$\therefore$ If two planes cut, &c.

STRAIGHT LINES PERPENDICULAR TO PLANES.

DEFINITION.

A straight line is said to be **perpendicular to a plane** when it is perpendicular to every straight line in that plane which meets it.

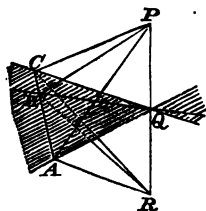
PROPOSITION VI.

If a straight line be perpendicular to each of two straight lines at their point of intersection, it shall be perpendicular to the plane passing through them.

Let PQ be $\perp$ to QA and QC .

Then shall PQ be $\perp$ to the plane passing through QA, QC .

Through Q in the same plane as QA, QC draw any other straight line QH ; and through any point H in QH draw a straight line cutting QA, QC in A and C ; produce PQ making $QR = PQ$; join PA, PC, PH, RA, RC, RH .



Then $\therefore PQ, QA$ and right $\angle PQA$

are respectively $= RQ, QA$ and $\angle RQA$;

$$\therefore PA = RA. \quad (I. 1)$$

Similarly $PC = RC$,

and AC is common to $\triangle s PCA, RCA$;

$$\therefore \angle PCA = \angle RCA; \quad (I. 5)$$

also PC, CH are respectively $= RC, CH$;

$$\therefore PH = RH; \quad (I. 1)$$

also PQ, QH are respectively $= RQ, QH$;

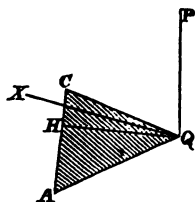
$$\therefore \angle PQH = \angle RQH; \quad (I. 5)$$

$$\therefore PQ \text{ is } \perp \text{ to } QH;$$

$\therefore PQ$ is $\perp$ to the plane through QA, QC . (Def.)

Conversely :

If a straight line is perpendicular to each of three straight lines which meet in a point, these three straight lines are in one and the same plane.



Let PQ be $\perp$ to QA, QC, QX .

Then shall these three straight lines be in one and the same plane.

For if not, if possible, let QX be out of the plane in which are QA, QC , and let the plane in which are PQ, QX cut the former in QH .

Then PQ is $\perp$ to QH , (by the Prop.)

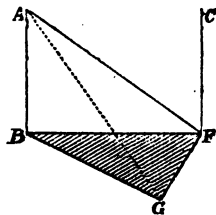
but it is also $\perp$ to QX ; (hyp.)

which is impossible.

$\therefore$ If a straight line, &c.

PROPOSITION VII.

If two straight lines be at right angles to the same plane, they shall be parallel to one another.



Let AB , CF be each $\perp$ to the plane BFG .

Then shall AB be $\parallel$ to CF .

Join BF and draw FG in the plane $BFG \perp$ to BF and $= AB$.

Join BG , AG , AF .

Then $\because AB$ is $\perp$ to the plane BFG , $\therefore \angle ABF$ is a right $\angle$;

$\therefore AB$, BF and $\angle ABF$ are respectively $= GF$, FB , and $\angle GFB$;

$$\therefore AF = GB. \quad (1.1)$$

Hence AF , FG , GA are respectively $= GB$, BA , AG ;

$$\therefore \angle AFG = \angle GBA. \quad (1.5)$$

But $\angle GBA$ is a right $\angle$, $\therefore AB$ is $\perp$ to plane BFG ;

$$\therefore \angle AFG \text{ is a right } \angle;$$

also $\angle CFG$ is a right $\angle$, $\therefore CF$ is $\perp$ to plane BFG ;

$\therefore GF$ is $\perp$ to each of the three straight lines FB , FA , FC ;

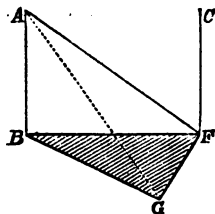
$\therefore FC$ is in the same plane as AF , FB , in which plane is also AB . (Prop. 6)

Then $\because ABF$, CFB are right $\angle$ s;

$$\therefore AB \text{ is } \parallel \text{ to } CF. \quad (1.20)$$

Conversely:

If two straight lines be parallel, and one of them be at right angles to a plane, the other shall also be at right angles to the same plane.



Let AB be $\parallel$ to CF , and CF be $\perp$ to the plane BFG .

Then shall AB be $\perp$ to the plane BFG .

Join BF , and in the plane BFG draw $GF \perp$ to BF
and $= AB$.

Join BG, AG, AF .

Then $\because CF$ is $\perp$ plane BFG , $\therefore \angle$ s CFB, CFG are right $\angle$ s;

and $\because AB$ is $\parallel$ to CF , $\therefore \angle$ s ABF, CFB are together = two right $\angle$ s; $\therefore \angle ABF$ is a right $\angle$, (I. 21)

and $\because GF$ is $\perp$ to CF , and also $\perp$ to BF ;

$\therefore GF$ is $\perp$ to the plane in which are CF, BF (Prop. 6),
and $\therefore \perp$ to AF , $\therefore \angle AFG$ is a right $\angle$.

Again, $\because AB, BF$ and $\angle ABF$ are respectively = GF, FB and $\angle GFB$;

$\therefore AF = GB$. (I. 1)

Hence AF, FG, GA are respectively = GB, BA, AG ;

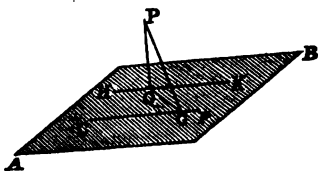
$\therefore \angle AFG = \angle ABG$; (I. 5)

$\therefore \angle ABG$ is a right $\angle$, and also $\angle ABF$ is a right $\angle$;

$\therefore AB$ is $\perp$ to the plane BFG . (Prop

PROPOSITION VIII.

It is always possible to draw a straight line perpendicular to a plane from a given point without it.



For let AB be a given plane, and P a point without it, and let it be required to draw from P a straight line $\perp$ to the plane AB .

In the plane AB draw any straight line CF , and from P let fall $PG \perp$ to CF .

Then if PG is also $\perp$ to the plane AB , the thing required has been done.

But, if not, in the plane AB draw $GQ \perp$ to CF , and from P let fall $PQ \perp$ to GQ ; then PQ is $\perp$ to the plane AB .

Through Q draw $HK \parallel$ to CF .

Then $\because CF$ is $\perp$ PG and also $\perp$ QG ;

$\therefore CF$ is $\perp$ to plane PGQ . (Prop. 6)

HK is $\parallel$ to CF , $\therefore HK$ is $\perp$ to plane PGQ , (Prop. 7)

$\perp$ to PQ ;

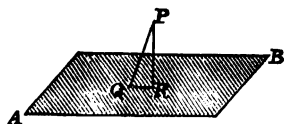
Q is $\perp$ to HK , and PQ is also $\perp$ to QG ;

$\therefore PQ$ is $\perp$ to plane AB . (Prop. 6)

is a

PROPOSITION IX.

There cannot be drawn more than one straight line perpendicular to a plane from a given point without it.



For, if possible, let PQ , PR be each of them $\perp$ to the plane AB .

Join QR .

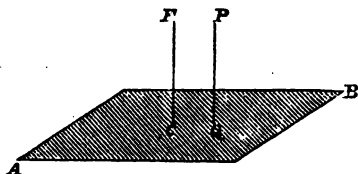
Then PQR , PRQ are each of them right angles ;

which is impossible.

(I. 13)

PROPOSITION X.

It is always possible to draw a straight line perpendicular to a plane from a given point within it.



For let AB be a given plane, and C a point within it, and let it be required to draw from C a straight line $\perp$ to the plane AB .

From any point P without the plane let PQ be drawn $\perp$ to the plane.

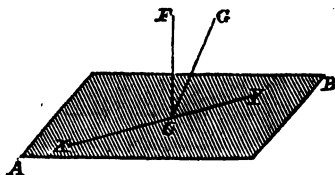
Through C draw $CF \parallel$ to PQ .

Then shall CF be $\perp$ to the plane AB .

For if two straight lines are $\parallel$ and one of them is $\perp$ to a plane, the other is also $\perp$ to the same plane. (Prop. 7)

PROPOSITION XI.

There cannot be drawn more than one straight line perpendicular to a plane from a given point within it.



For, if it be possible, from a point C within the plane AB , let two straight lines CF , CG be drawn $\perp$ to the plane AB , and let the plane through CF , CG cut the plane AB in XY .

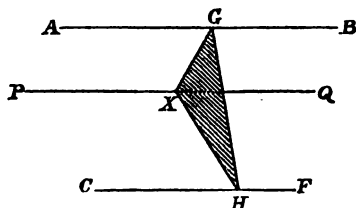
Then $\angle s$ FCY , GCY are each right $\angle s$, and are $\therefore$ equal to one another ; which is impossible.

PARALLEL LINES

(not all situated in the same plane).

PROPOSITION XII.

Two straight lines which are each of them parallel to the same straight line, which is not in the same plane with them, are parallel to one another.



Let AB , CF be each of them $\parallel$ to PQ
 (AB , CF , PQ not being all in the same plane).

Then shall AB be $\parallel$ to CF .

In PQ take any point X ,
 and in the plane in which are AB , PQ draw $XG \perp$ to PQ ;
 also in the plane in which are CF , PQ draw $XH \perp$ to PQ .

Then since PQ is $\perp$ to XG and XH ;

$\therefore PQ$ is $\perp$ to the plane GXH , (Prop. 6)

but AB is $\parallel$ to PQ , $\therefore AB$ is $\perp$ to the plane GXH .

(Prop. 7)

Similarly CF is $\perp$ to the plane GXH ;

$\therefore AB$ is $\parallel$ to CF . (Prop. 7)

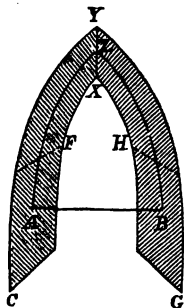
PARALLEL PLANES.

DEFINITION.

Planes which never meet, however far they are produced in any direction, are called **parallel planes**.

PROPOSITION XIII.

Planes to which the same straight line is perpendicular are parallel to one another.



Let the straight line AB be $\perp$ to each of the planes CF, GH .

Then shall CF be $\parallel GH$.

For if not, if possible, let the planes CF, GH be produced to meet in YX .

In YX take any point Z ; join AZ, BZ .

Then $\because AB$ is $\perp$ to the plane CF , $\therefore \angle BAZ$ is a right $\angle$;

so $\angle ABZ$ is a right $\angle$.

Hence in $\triangle ABZ$ two of the $\angle$ s are right $\angle$ s;

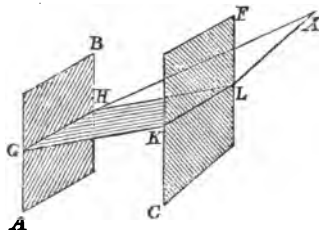
which is impossible; (I. 13)

$\therefore$ planes CF, GH will never meet however far they are produced;

$\therefore CF, GH$ are parallel.

PROPOSITION XIV.

If two parallel planes are cut by another plane, their common sections with it are parallel.



Let the two parallel planes AB , CF be cut by the plane HK .

Then shall their common sections GH , KL be parallel.

For if not, if possible, let them meet when produced in X .

Then since GH is in plane AB , $\therefore X$ is in plane AB ;
(Prop. I)

so X is in plane CF ;

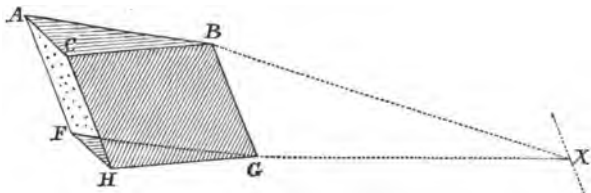
$\therefore$ planes AB , CF will meet if produced;
which is impossible, since they are parallel;

$\therefore GH$, KL will not meet if produced;

$\therefore GH$, KL are parallel.

DEDUCTION (A).

If two straight lines meeting one another be parallel to two others that meet one another, and are not in the same plane with the first two, the first two and the other two shall contain equal angles, also the plane through the first two shall be parallel to the plane through the other two.



Let AB, BC two straight lines meeting one another be $\parallel$ to FG, GH , meeting one another, the latter two straight lines not being in the same plane with the former.

Then shall $\angle ABC$ be $= \angle FGH$,
and the plane ABC be $\parallel$ to plane FGH .

Cut off $GF = BA$ and $GH = BC$; join AC, FH, AF, CH, BG .

Then $\therefore GF$ is $=$ and $\parallel$ to AB , $\therefore AF$ is $=$ and $\parallel$ to BG ,

(I. 23)

so also is $CH =$ and $\parallel$ to BG ;

(I. 23)

$\therefore AF$ is $=$ and $\parallel$ to CH ;

(Prop. 12)

$\therefore CA$ is $=$ and $\parallel$ to HF .

(I. 23)

Then $\therefore AB, BC, CA$ are respectively $= FG, GH, HF$;

$\therefore \angle ABC = \angle FGH$.

(I. 5)

Also plane ABC shall be $\parallel$ to plane FGH ; for if not, if possible, let them meet when produced, then their common section will be a straight line.

(Prop. 5)

Since AB, BC cannot both be $\parallel$ to the common section one of them will meet it if produced.

Let AB produced meet it in X .

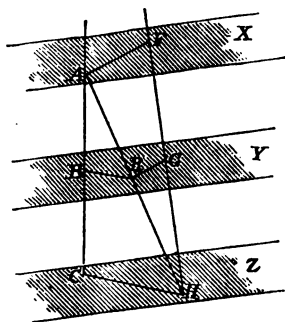
Thus X is in the plane containing the parallels AB, FG and also in the plane FGH ; and is $\therefore$ in their common section of which FG is a part.

Hence AB, FG meet if produced; which is impossible.

$\therefore$ plane ABC is $\parallel$ to plane FGH .

DEDUCTION (B).

If two straight lines be cut by parallel planes, they shall be cut proportionally.



Let the two straight lines AC , FH be cut by the parallel planes X , Y , Z in A , B , C and F , G , H respectively.

Then $AB : BC$ as $FG : GH$.

Join AH cutting the plane Y in the point R .

Join AF , CH , BR , RG .

Then $\therefore$ the parallel planes Y , Z are cut by the plane AH ,

$\therefore$ their common sections BR , CH are $\parallel$ to one another;

(Prop. 14)

$\therefore AB : BC$ as $AR : RH$. (vi. 1)

Similarly it may be shewn that

$FG : GH$ as $AR : RH$;

$\therefore AB : BC$ as $FG : GH$. (v. 3)

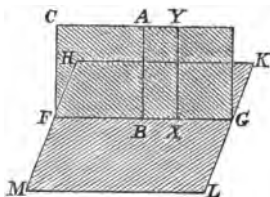
PLANES PERPENDICULAR TO PLANES.

DEFINITION.

One plane is said to be **perpendicular to another plane** when every straight line drawn in one of the planes perpendicular to the common section of the two planes is perpendicular to the other plane.

PROPOSITION XV.

If a straight line be perpendicular to a plane, every plane which passes through it shall be perpendicular to that plane.



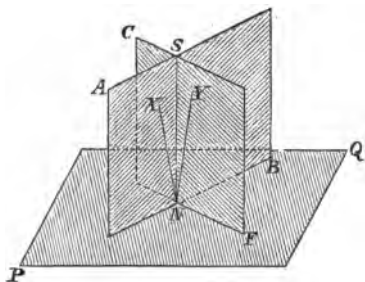
Let AB be a straight line $\perp$ to the plane $HKLM$,
and CFG a plane passing through AB .
Then shall CFG be $\perp$ to the plane HKL .

In the common section FG take any point X , and
through X in the plane CFG draw $XY \perp$ to FG .

Then $\because AB$ is $\perp$ to plane HKL , $\therefore AB$ is $\perp$ to FG ;
 $\therefore \angle ABX$ is a right $\angle$;
 also $\angle YXB$ is a right $\angle$; (constr.)
 $\therefore YX$ is $\parallel$ to AB ; (I. 20)
 but AB is $\perp$ to plane HKL , (hyp.)
 $\therefore YX$ is $\perp$ to plane HKL ; (Prop. 7)
 $\therefore$ the plane CFG is $\perp$ to the plane HKL . (Def.)

PROPOSITION XVI.

If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.



Let the two planes ANB , CNF which cut one another be each $\perp$ to the plane PQ .

Then shall their common section SN be $\perp$ to the plane PQ .

For, if not, if possible, in plane ANB draw $NX \perp$ to NB , and $\therefore \perp$ to the plane PQ , and in the plane CNF draw $NY \perp$ to NF , and $\therefore \perp$ to plane PQ ,

$\therefore$ from the point N two straight lines have been drawn $\perp$ to the plane PQ ; which is impossible; (Prop. 11)

$\therefore SN$ is $\perp$ to plane PQ .

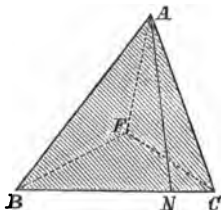
SOLID ANGLES.

DEFINITION.

A solid angle is that which is made by the meeting in one point of two or more plane angles, which are not in the same plane.

PROPOSITION XVII.

If a solid angle be contained by three plane angles any two of them are together greater than the third.



Let the solid angle at A be contained by the three plane angles BAF , FAC , CAB .

Then shall any two of them BAF , FAC be together greater than the third CAB .

If either of the two angles BAF , FAC be $=$ or $>$ $\angle CAB$, the proposition is evidently true.

But, if not, from $\angle CAB$ cut off $\angle BAN = \angle BAF$;

make $AN = AF$; through N draw BNC meeting AB , AC in B and C ; join BF and CF .

Then $\therefore BA, AF$ and $\angle BAF$ are respectively $= BA, AN$ and $\angle BAN$,

$$\therefore BF = BN. \quad (I. 1)$$

Again $\therefore BF, FC$ are together $> BC$ and BF is $= BN$;

$$\therefore FC > NC.$$

Also AF, AC are respectively $= AN, AC$;

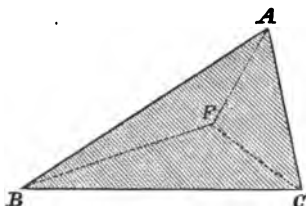
$$\therefore \angle FAC > \angle NAC; \quad (I. 19)$$

$\therefore \angle s BAF, FAC$ are together $> \angle s BAN, NAC$,

$$\text{i.e. } > \angle CAB.$$

PROPOSITION XVIII.

The plane angles which contain any solid angle are together less than four right angles.



First let the solid angle at A be contained by three plane angles BAF, FAC, CAB .

Then shall these angles be together less than four right $\angle$ s.

In AB, AC, AF take any points B, C, F ,
and join BC, CF, FB .

Then since the solid $\angle$ at B is contained by three plane angles;

$\therefore \angle$ s ABC, ABF are together $> \angle CBF$.

(Prop. 17)

Similarly $\angle$ s ACB, ACF are together $> \angle FCB$,

(Prop. 17)

and $\angle$ s $AFC, AFB > \angle BFC$; (Prop. 17)

$\therefore \angle$ s $ABC, ABF, ACB, ACF, AFC, AFB$ are together $>$ the $\angle$ s of $\triangle BCF$, i.e. the $\angle$ s at the bases of the $\triangle$ s ABC, ACF, AFB are together $>$ two right $\angle$ s.

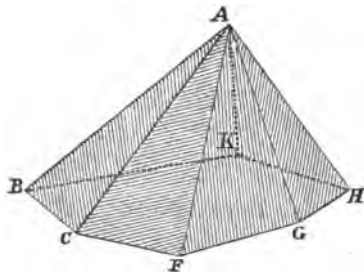
(I. 24)

Now all the $\angle$ s of the $\triangle$ s ABC, ACF, AFB are = six right $\angle$ s;

(I. 24)

$\therefore$ the $\angle$ s at the vertex A are $<$ four right $\angle$ s.

Next, let the solid angle at A be contained by any number of plane angles.



Let AB , AC , &c. be cut by a plane in the points B , C , F , &c. and join BC , CF , FG , &c.

Then $\angle$ s ACB , ACF are together $> \angle BCF$,
(Prop. 17)

so $\angle$ s AFC , AFG are together $> \angle CFG$: and so on.

$\therefore$ all the $\angle$ s at the bases of the Δ s ABC , ACF , &c. are together $>$ the angles of the polygon BCF .

Now all the $\angle$ s of the Δ s ABC , ACF , &c. are = twice as many right $\angle$ s as there are Δ s,
(I. 24)

and all the $\angle$ s of the polygon BCF together with four right $\angle$ s are equal to twice as many right $\angle$ s as the figure has sides;
(I. 30)

$\therefore$ all the $\angle$ s of the Δ s ABC , ACF , &c. are = the $\angle$ s of the polygon together with four right $\angle$ s;

but the $\angle$ s at the bases of the Δ s ABC , ACF , &c. are $>$ the $\angle$ s of the polygon;

$\therefore$ the $\angle$ s at the vertex A are $<$ four right $\angle$ s.

GEOMETRICAL EXERCISES.

DEFINITIONS.

One angle is called the **complement** of another, when the two together make up a right angle.

One angle is called the **supplement** of another, when the two together make up two right angles.

When two straight lines are drawn from a point, they form an angle less than two right angles, and also an angle greater than two right angles. The latter is called a **reflex angle**, and together with the former makes up four right angles.

When two sides of a polygon form a **re-entrant angle**, the reflex angle, not the smaller angle, is considered as one of the angles of the polygon.

A **scalene triangle** is one contained by three unequal straight lines.

An **oblong** has all its angles right angles, but has not all its sides equal.

A **rhombus** has all its sides equal, but its angles are not right angles.

A **rhomboid** has its opposite sides equal to each other, but all its sides are not equal nor its angles right angles.

A **trapezoid** is a four-sided figure which has two of its sides parallel, and the remaining two not parallel.

N.B. This figure is sometimes called a **trapezium**; but Euclid calls all quadrilateral figures trapeziums which are not parallelograms.

If a straight line AB be divided in P and Q so that

$$AP : AB \text{ as } PQ : QB,$$

then AB is said to be divided **harmonically**.

Figures whose areas are equal are said to be **equivalent** to one another.

A **tetrahedron** is a solid contained by four planes.

A **polyhedron** is a solid contained by more than four planes.

A **dihedral angle** is formed by the intersection of two planes.

PLANE GEOMETRY.

BOOK I.

1. If on the same base and on the same side of it two isosceles triangles are drawn, the vertex of one triangle must fall within the other.

2. If on the same base and on opposite sides of it two isosceles triangles are drawn, the straight line joining their vertices shall bisect the base.

3. Let the equal sides AB , AC of an isosceles triangle be produced to F , G making $AF = AG$: join FC , BG intersecting in H , then AH will bisect the angle BAC .

4. Prove that in the previous figure the lines bisecting the $\angle$ s at F , G will meet AH in the same point.

5. If the angles of one triangle are not severally equal to those of another, neither are their sides equal.

6. Divide a given angle into four equal angles.

7. Describe a right-angled triangle whose hypotenuse shall be equal to one straight line, and one of its sides equal to another straight line.

8. Draw a straight line through a given point cutting off equal parts from the arms of a given angle.

9. Find a point in a straight line equidistant from two given points without it.

10. Trisect a given right angle.

11. Construct a triangle having given the base, one of the angles at the base, and the sum of the sides.

12. Draw a straight line through a given point equally inclined to two given straight lines.

13. Find a point in the base of a triangle equidistant from the two sides.

14. Describe an isosceles $\triangle$ having each of the sides double the base.

15. Describe an isosceles triangle having each of the sides three times the size of the base.

16. Find a point in a straight line at a given distance from a given point.

17. Find a point in the circumference of a circle at a given distance from a given point.

18. The lines drawn from the angular points of a Δ through any point within it to the opposite sides are together greater than the semi-perimeter of the Δ .

19. The difference between any two sides of a triangle is less than the third side.

20. Find the shortest path from a given point in one of two straight lines to the other, and then back again to the former straight line.

21. Given the base and one of the sides of an isosceles triangle, construct it.

22. Prove that the sum of the distances of any point from the three angles of a triangle is greater than half the perimeter.

23. Determine the shortest path from one point to another subject to the condition that it shall meet a given straight line.

24. Determine the shortest path from one point to another subject to the condition that it shall meet two given straight lines.

25. The four sides of a quadrilateral figure are together greater than the two diagonals.

26. Prove that the straight lines drawn through the extremities of the base of an isosceles triangle equally inclined to it and terminated by the sides, are equal to one another.

27. If straight lines be drawn to the angular points of a triangle from a point within it, then of the three angles thus formed there cannot be more than one acute angle.

28. Trisect a given straight line.

29. Describe a triangle equiangular to a given triangle and having its perimeter equal to a given straight line.

30. Find a point situated at a given distance from a given point, and also at a given distance from a given straight line. Is the problem always possible?

31. Through two given points draw two straight lines forming with a given straight line an equilateral triangle.

32. Through two given points draw two straight lines forming with a given straight line a triangle equiangular to a given triangle.

33. Describe a rhombus having each of one pair of opposite angles double each of the other pair.

34. If the two diameters of a parallelogram be drawn, it will be divided into four equal parts.

35. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

36. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

37. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

38. Draw lines through the angular points of a parallelogram which shall form another parallelogram double the former.

39. If equal triangles are described on the same base on opposite sides of it, the line joining their vertices is bisected by the base or base produced.

40. AC, BD are equal straight lines drawn from the extremities of the straight line AB , on opposite sides of it, such that the $\angle s$ BAC, ABD are together equal to two right angles, prove that AB bisects the straight line joining CD .

41. If any point P be taken within a parallelogram $ABCD$, then $\triangle ABP + \triangle CDP = \triangle BCP + \triangle DAP$.

42. The perimeter of an isosceles triangle is less than that of any other equal triangle on the same base.

43. If an angle of a parallelogram is bisected by a diagonal, then the opposite angle is also bisected by it.

44. If one angle of a parallelogram is bisected by a diagonal, then all the angles are bisected by one or other of the diagonals.

45. If an angle of a parallelogram is bisected by a diagonal, the figure is a rhombus.

46. Any three angles of a convex pentagon with no re-entrant $\angle s$ are together greater than the angles of a triangle.

47. Any four angles of a convex hexagon with no re-entrant angles are together greater than the angles of a quadrilateral.

48. Any angles of a convex rectilineal figure are together greater than all the angles of a figure of which the number of sides is the same as the number of angles of the first figure taken (there being no re-entrant $\angle s$).

49. A quadrilateral figure is divided into four equal triangles by straight lines drawn from a point within the

figure to the angular points; prove that the figure is a parallelogram.

50. Divide a parallelogram into four equal parts by straight lines drawn from a given point in one of its sides.

51. $ABDC$ is a parallelogram; P is a point without the angle BAC ; prove that $\triangle PAD = \triangle PAB + \triangle PAC$.

52. $ABDC$ is a parallelogram; P is a point within the angle BAC ; prove that $\triangle PAD =$ the difference between the $\triangle$ s PAB, PAC .

53. Two triangles having equal bases are together equal to a triangle having a base equal to that of either of the triangles, and altitude equal to the sum of their altitudes.

54. If one diagonal of a quadrilateral figure bisect the other, it bisects the figure also.

55. If a quadrilateral, two of whose sides are parallel, be bisected by each of two straight lines drawn from the extremities of the shortest of those parallel sides, then these bisectors bisect each other.

56. If a quadrilateral figure be bisected by a straight line which also bisects its opposite sides, then those opposite sides shall be parallel.

57. The area of a rhombus is equal to half the rectangle contained by the diagonals.

58. Draw an isosceles triangle having each of the angles at the base half as large again as the vertical angle.

59. Bisect a parallelogram by a straight line passing through a given point.

60. Prove that the diagonals of a rhombus bisect one another at right angles.

61. If the opposite angles of a rhombus are together equal to two right angles, the figure is a square.

62. Bisect a triangle by a straight line drawn from a given point in one of its sides.

63. Bisect a quadrilateral by a straight line through one of its angular points.

64. Find a point in a straight line whose distance from another given point in that line shall be equal to its distance from another given straight line.

65. Draw a straight line through a given point which shall make a given angle with a given straight line. Shew that there are two solutions if the given angle is either obtuse or acute.

66. Describe a parallelogram equal in area and perimeter to a given triangle.

67. Describe a triangle equal to a given triangle and having two of its sides respectively equal to two given straight lines. When is the problem impossible?

68. In the base of a triangle find a point from which lines drawn $\parallel$ to the sides are equal.

69. Given two sides of a triangle, find the third side so that the area of the triangle may be the greatest possible.

70. Find a point in the side or side produced of a parallelogram, such that the angle which that side makes with the line joining the point with one extremity of the

opposite side may be bisected by the line joining the point and the other extremity.

71. Inscribe an equilateral triangle in a square, having one of its angular points at one of the corners of the square.

72. Inscribe an equilateral triangle in a square such that one of its angular points shall be at the middle point of one of the sides of the square.

73. Through a given point within a given angle draw a straight line which shall be bisected at that point.

74. The area of a quadrilateral figure whose base is parallel to the opposite side is half that of a parallelogram having the same altitude and base equal to the sum of the parallel sides of the given figure.

75. If one angle of a triangle be equal to the other two together, the greatest side is double of the distance of its middle point from the opposite angle.

76. If each of the equal angles of an isosceles triangle be one-fourth of the third angle, and from one of them a perpendicular be drawn to the base meeting the opposite side produced; then will the part produced, the perpendicular and the remaining side, form an equilateral triangle.

77. Describe a rectangle equal to a given rectangle and having one of its sides equal to a given straight line.

78. Prove that all the interior angles of an octagon are together double all the interior angles of a pentagon.

79. If one quadrilateral figure be equiangular to another and have two adjacent sides equal to the corre-

sponding sides of the other, the figure shall be equal in all respects.

80. Two broken lines are composed of straight lines : also the lines and angles of one are respectively equal to the lines and angles of the other ; the straight line joining the extremities of the one shall be equal to that joining the extremities of the other.

81. If all the sides and angles of one polygon be given equal to the corresponding sides and angles of another polygon with the exception of one angle and the sides containing it in each respectively, then shall these also be equal.

82. If all the sides and angles of one polygon be given equal to the corresponding sides and angles of another polygon, with the exception of one side in each and the angles at the extremities of those sides, then shall the polygons be equal in all respects.

83. Produce the sides of a given heptagon both ways till they meet forming seven triangles : required the sum of their vertical angles.

84. The sides of a polygon are produced both ways till they meet ; find the sum of the vertical angles formed.

85. Describe a square which shall be double of a given square.

86. Describe a square whose area shall be three times that of a given square.

87. Describe two squares equal to a given square : one of them being three times that of the other.

88. Describe a square which shall have the arms of one of its angles falling upon the sides of a given right-angled triangle and the vertex of the opposite angle on the hypotenuse.

89. Find a square equal to the difference between two given squares.

90. From AC the diagonal of a square $ABCD$, cut off AE equal to one-fourth of AC , and join BE , DE . Shew that the figure $BADE$ equals twice the square on AE .

91. Squares are described on the three sides of a right-angled triangle, and the adjacent angular points of the squares joined; prove that each of the triangles so formed equals the given triangle.

92. Squares are described on the sides of a right-angled triangle, and the adjacent angular points of the squares joined. Prove that the area of the hexagon thus formed is equal to the square on the hypotenuse together with the square of a line equal to the sum of the other two sides.

93. If the diagonals of a quadrilateral cut each other at right angles, prove that the squares described on two opposite sides will be equal to the squares on the other two sides.

94. If ACB be any triangle, and CD be drawn $\perp$ to AB , prove that the difference between the squares on AD , DB is equal to the difference between the squares on AC , CB .

BOOK II.

1. The sum of the perpendiculars let fall on the sides of an equilateral figure from any point within it is constant.

2. Divide a straight line into two parts, so that the sum of their squares may be the least possible.

3. If a straight line be drawn from the vertex of an isosceles triangle to any point in the base, the square on this line together with the rectangle under the segments of the base equals the square of either of the other sides.

4. Divide a straight line into two parts so that the rectangle contained by the whole and one of the parts may be equal to the rectangle contained by the other part and a given straight line.

5. Divide a straight line into two parts so that the rectangle contained by the segments shall be the greatest possible.

6. Divide a straight line into two parts so that the difference of the squares on the parts is equal to twice the rectangle contained by the parts.

7. If from one of the equal angles of an isosceles triangle, a perpendicular be drawn to the opposite side, the rectangle contained by that side and the segment of it intercepted between the perpendicular and base is equal to half the square on the base.

8. If ABC be an isosceles triangle, and DE be drawn parallel to the base BC and EB joined: prove that

$$\text{square on } BE = \text{rect. } (BC, DE) + \text{square on } CE.$$

9. Divide a straight line into two parts so that the squares on the whole line and on one of the parts shall be together equal to three times the square on the other part.

10. Prove that the squares on the diagonals of a trapezoid are together equal to the squares on its two sides which are not parallel and twice the rectangle contained by the sides which are parallel.

11. Divide a straight line into two parts so that the rectangle contained by them may be equal to a given rectangle.

12. Describe a rectangle equal to a given square, and having its perimeter equal to a given straight line.

13. In any triangle ABC , if BP , CQ be drawn respectively perpendicular to CA and BA , produced if necessary; shew that

square on BC = rect. (BA, BQ) + rect. (CA, CP) .

14. The squares on two sides of a triangle are together double of the squares on the straight line drawn from the vertex to the middle point of the base and on half the base.

15. Given the lengths of the three lines drawn from the angles of a triangle to the points of bisection of the opposite sides, construct the triangle.

16. Describe a rectangle equal to a given square, and also having the difference between its adjacent sides equal to a given straight line.

17. If a perpendicular be let fall from the vertex of a triangle upon the base, then will the difference between

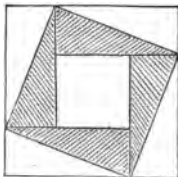
the squares on the sides be equal to the rectangle contained by the base and the difference between the segments of the base.

18. The square on the hypotenuse of a right-angled triangle is $<$ the square on a straight line which is $=$ the sum of its sides by twice the rect. contained by those sides. Hence deduce the theorem of Pythagoras.

19. The square on the hypotenuse of a right-angled triangle is $>$ the square on a straight line which is $=$ the difference of its sides by twice the rectangle contained by those sides.

20. The square on the sum of the sides of a right-angled triangle together with the square on their difference is double of the square on the hypotenuse.

N.B. Exercises (18), (19), (20) may be easily established by means of the accompanying figure.



21. In the straight line RQ take any point S : bisect SQ in T , and produce QR making $RP = QR$. Prove that $\text{rect. } PT, RS + 5 \text{ rect. } RQ, QT = 2 \text{ sq. on } RQ + 2 \text{ sq. on } QT$.

22. The squares on the lines drawn from any given point to the extremities of one diagonal of a rectangle are

together equal to the squares on the lines drawn to the extremities of the other diagonal.

23. The base of a triangle is fixed, and the sum of the squares on the sides is constant ; determine the locus of the vertex.

24. In every parallelogram the squares on the sides are together equal to the squares on the diagonals.

25. What triangles are those whose sides are
(13, 4, 12) (13, 5, 12) (13, 6, 12) (13, 6, 7) (13, 6, 6)?

26. What triangles are those whose sides are
(12, 10, 8) (10, 8, 6) (8, 6, 4) (6, 4, 2)?

27. Produce a straight line so that the rectangle contained by the whole line and the part produced shall be equal to the square on the given straight line.

28. Produce a straight line so that the rectangle contained by the given line and the part produced may be equal to a given triangle.

29. Draw two squares equal to a given square : one of them being double of the other.

30. Draw a square which shall be equal to two given triangles together.

31. The perimeter of an equilateral triangle is greater than that of an equal square.

32. Produce a given straight line so that the square on the whole line thus produced may be double the square on the part produced.

33. Of all equal rectangles the square has the least perimeter.

BOOK III.

1. Divide a circle into three equal parts.
2. Through a given point within a circle draw the least possible chord.
3. Through a given point within a circle draw a chord which shall be bisected in that point.
4. Bisect a circle by a straight line drawn through a given point without it.
5. If two chords be drawn in a circle through any point in a diameter equally inclined to it, they shall be equal to one another.
6. A chord PQ of a circle is cut by any diameter AB ; prove that the difference between the areas of the triangles PAQ , PBQ is constant.
7. If from a point within a circle two straight lines are drawn to the circumference equal to one another, the centre lies in the straight line bisecting the angle between them.
8. If from a point without a circle two straight lines are drawn to the circumference equal to one another, the centre lies in the straight line bisecting the angle between them.
9. Describe a circle passing through two given points and bisecting the circumference of a given circle.
10. Draw a tangent to a circle parallel to a given straight line.

11. Draw a straight line touching a given circle and making a given angle with a given straight line.

12. ABC is a triangle, having an acute angle at A ; shew that the square on BC is $<$ the squares on BA, AC by twice the square on a line drawn from A to touch the circle described on BC as diameter.

13. Find a point such that the tangents drawn from it to two given circles may contain a given angle.

14. Find a point from which if straight lines be drawn to touch three given circles, neither of which lies within the other, the tangents so drawn shall be equal.

15. Find the locus of points from which the tangents drawn to two given circles are equal.

16. Describe a circle passing through one of the angular points of a square, and touching the sides containing the opposite angle.

17. Through a given point draw a straight line cutting a given circle so that the intercepted chord shall be equal to a given straight line.

18. Find the locus of the centre of a circle which touches two intersecting straight lines.

19. Find the locus of the centre of a circle which touches two parallel straight lines.

20. Describe about a circle the least possible parallelogram.

21. Inscribe in a circle the greatest possible rectangle.

22. On the base of a segment of a circle, describe the

greatest possible triangle having its vertex in the circumference.

23. Find a point such that the tangents drawn from it to two given circles shall be equal to two given straight lines.

24. In a given circle draw a diameter at a given distance from a given point. State the limits of the problem.

25. If two straight lines touch a circle, the straight line bisecting the angle between them shall pass through the centre.

26. Describe a circle equal to a given circle, and touching two given straight lines. Is this problem always possible?

27. A circle touches two sides AB , AC of a triangle in F and E . Also BC is equal to BF and CE together: prove that BC touches the circle.

28. If a tangent be drawn to a circle from any point in the circumference equal to the radius, prove that the line joining its extremity with the centre is equal to the parallel chord drawn through the point of contact.

29. If two circles touch each other externally, describe a circle which shall touch one of them in a given point, and also touch the other. In what case does this become impossible?

30. Let one circle touch another internally, and let straight lines touch the inner circle and be terminated by the outer; shew that the greatest of these lines is the one parallel to the common tangent at the point of contact.

31. If two circles touch each other externally and any third circle touch both, prove that the difference of the distance of the centre of the third circle from the centres of the other two is constant.

32. Describe a circle passing through a given point and touching a given circle in a given point.

33. Describe a circle passing through two given points and touching a given circle.

34. Describe a circle touching a given straight line in a given point, and also a given circle.

35. If three circles touch each other externally, prove that the tangents at their points of contact pass through one point.

36. Draw lines from two given points which shall meet in a given straight line and contain a right angle. Within what limits is the problem possible?

37. In a given segment inscribe a similar segment.

38. Divide a circle into two segments such that the angle in one of them shall be double of the angle in the other.

39. Divide a circle into two segments such that the angle in one of them shall be three times the angle in the other.

40. Divide a circle into two segments such that the angle in one of them shall be five times that in the other.

41. Given the centre of a circle: describe a second circle whose radius shall be equal to the diameter of the first by means of the compasses alone.

42. Of all triangles on the same base and having their vertical angles equal to a given angle, find that which has the greatest area.

43. Parallel chords in a circle intercept equal arcs.

44. The straight lines joining the extremities of equal arcs in a circle will form a quadrilateral, two of whose opposite sides are parallel, the other two being equal, and its diagonals equal.

45. If a circle be described passing through the opposite angles of a parallelogram and cutting the four sides, and the points of intersection joined so as to form a hexagon, the straight lines thus drawn shall be parallel.

46. If two chords of a circle intersect at right angles, the opposite segments are together equal to a semicircle.

47. If two equal chords of a circle intersect at right angles, two of the segments are quadrants.

48. The diameter of a circle cuts a chord at an angle equal to half a right angle. Prove that the sum of the squares of the segments of the chord is equal to half the square on the diameter.

49. If the circumference of one circle passes through the centre of another, the lines joining the points of intersection to any point in the aforesaid circumference shall be equally inclined to the straight line joining that point and the aforesaid centre.

50. If the circumference of one circle passes through the centre of another, any two chords of the second drawn from the points of intersection so as to cut one another in the aforesaid circumference shall be equal.

51. If from any point in the arc of a sector perpendiculars be let fall on the bounding radii, the distance between their feet will be constant.

52. If on two straight lines containing an angle, segments of circles be described containing angles equal to it, and falling without it, the lines produced will touch the segments.

53. From a given point without a circle, a straight line is drawn cutting the circle. Draw from the same point another line so as to intercept two arcs which together shall subtend at the centre an angle equal to a given angle.

54. If a circle be described about a triangle ABC , and perpendiculars be let fall from A, B, C on the opposite sides and produced to meet the circle in D, E, F , then will the arcs EF, FD, DE be bisected in A, B, C .

55. Given the vertical angle of a triangle, the sum of its arms, and also the difference of its arms; construct the triangle.

56. Find a point within a triangle such that, if straight lines be drawn from it to the three angles of a triangle, they shall make equal angles with one another. Is this problem always possible?

57. If two straight lines cut at right angles in a circle, the squares on the four segments are together equal to the square on the diameter.

58. If from two given points in the diameter of a circle, equidistant from the centre, straight lines be drawn to any point in the circumference the sum of their squares is constant.

59. If from any point without a circle lines be drawn cutting the circle and making equal angles with the longest line, they will cut off equal segments.

60. If from the angles of an equilateral triangle, inscribed in a circle, straight lines are drawn to any point in the circumference, then shall the two shorter lines thus drawn be equal to the longer one.

61. If any point be taken in the arc of a segment of a circle, cut off by one of the sides of an equilateral triangle inscribed in the circle, and straight lines drawn from it to the extremities of the base of the segment, then the sum of the squares on them together with the rectangle contained by them, shall be constant.

62. If from the angles of an equilateral triangle inscribed in a circle, straight lines are drawn to any point in the circumference, then shall the sum of the squares described upon them be constant.

63. The locus of the point, the sum of the squares of whose distances from the angular points of a given equilateral triangle is equal to twice the square on one of the sides, is the circumscribed circle.

64. Through a given point situated between two lines inclined at a given angle, to draw a line, having its extremities terminated in these lines such that the rectangle contained by the parts may be the least possible.

65. Draw lines from two given points meeting in a given straight line, and containing the greatest possible angle.

66. AB is the diameter of a circle, MN a chord parallel to AB , in AB take any point P , join PM , PN ,

and prove that the squares on PM , PN are together equal to the squares on AP , PB .

67. AB is the diameter of a circle, and tangents are drawn at the extremities A , B meeting a tangent at any point in D , F , prove that the angle subtended by FD at the centre is a right angle.

68. From a given point without a circle draw a straight line cutting the circle such that the intercepted chord shall be equal to the part without the circle.

69. Find the locus of the middle point of a rod which is moved so as always to have its extremities in the arms of a given right angle.

70. Describe an isosceles triangle having the angle at the vertex four times as large as each of the angles at the base.

71. Describe an isosceles triangle having the angle at the vertex treble each of the angles at the base.

72. Describe a right-angled triangle having one of the acute angles one-fourth of the other.

73. Describe a right-angled triangle having one of the acute angles half as large again as the other.

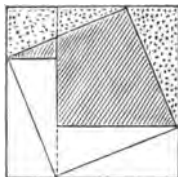
74. Describe a triangle having one of its angles half of another and treble the remaining angle.

BOOK IV.

1. Describe a circle touching one side of a triangle, and the other two sides produced.

2. On a given straight line describe a regular hexagon.
3. Find a point which is equally distant from three given points. Is this problem always possible?
4. Inscribe a circle in a rhombus.
Can a circle be described about a rhombus?
5. Describe a parallelogram about a circle, having an angle equal to a given angle.
6. If a parallelogram be described about a circle, it will be equilateral.
7. Determine the condition that the straight line joining the centres of the circles inscribed in, and described about a given triangle, may pass through one of the angular points of the triangle.
8. What relation exists between the sides of a triangle when the centres of the circles inscribed in and described about the triangle coincide?
9. Describe a regular hexagon having its angular points in the sides of a given equilateral triangle.
10. If the chords which bisect two angles of a triangle inscribed in a circle be equal, prove that either the angles are equal or else the third angle is equal to the angle of an equilateral triangle.
11. If one square be inscribed in another, the difference between their areas is equal to twice the rectangle contained by the segments of any one of the sides of the larger square.

12. If one square be inscribed in another, the area of the smaller square is equal to the squares on the segments of a side of the larger square.



13. Apply the previous deduction to shew how the square described on the hypotenuse of a right-angled triangle may be divided into parts which, when properly arranged, will form the squares on the sides containing the right angle.

14. If a quadrilateral figure be circumscribed about a circle, then two of its opposite sides are together equal to the other two.

15. If two of the opposite sides of a quadrilateral figure are equal to the other two, then a circle can be inscribed within it.

16. If a quadrilateral figure be circumscribed about a circle, then will the angles subtended at the centre by two of its opposite sides be together equal to two right angles.

17. Three circles touch each other externally, prove that their centres are the angular points of the triangle described about the circle which passes through the points of contact.

18. The circles each of which touches two sides of a regular pentagon at the extremity of a third meet in a point.

19. If two diagonals of a regular pentagon be drawn cutting one another, the greater segments will each be equal to a side of the pentagon.

20. An equilateral figure inscribed in a circle is also equiangular.

21. An equiangular figure is inscribed in a circle; prove that the alternate sides are equal.

22. An equiangular figure having an odd number of sides is inscribed in a circle: prove that it is equilateral.

23. On the base of a triangle describe an isosceles triangle whose vertical angle shall be double that of the given triangle. Is this problem always possible?

24. Describe a circle passing through one of the angular points of a rhombus, and touching the sides containing the opposite angle.

BOOK VI.

1. Through a given point draw a straight line which shall be divided in a given ratio by two given straight lines.

2. If perpendiculars be let fall from the extremities of one straight line upon another, prove that their feet are equidistant from the middle point of the former straight line.

3. Divide a straight line into two parts such that one of them shall be four times as long as the other.

4. Given the perimeter and the vertical angle of an isosceles triangle, construct it.

5. From two straight lines cut off two parts having a given ratio, so that the sum of the squares of the remainders may be equal to a given square.

6. If from any point in the circumference of the exterior of two concentric circles, two straight lines be drawn touching the interior and cutting the exterior, the distance between the points of contact will be half that between the points of intersection.

7. $PQRS$ is a square, PR its diagonal; bisect PS in T , and join QT cutting PR in K , then $RK =$ twice PK ; and $\triangle PTK : \triangle RTK : \triangle PQT : \triangle QRK$ as $1 : 2 : 3 : 4$.

8. ABC , ADE are two isosceles triangles having a common angle at A . Through B, E draw $BG, EF \parallel$ to one another; join DG, CF . Then shall DG be $\parallel$ to CF .

9. If RPS, RQS are two triangles on RS such that
 $RP : PS$ as $RQ : QS$,
 then the lines bisecting the angles at P, Q intersect RS in the same point.

10. A point is taken in one of the sides of a triangle, and through it a line is drawn $\parallel$ to another side meeting the third side, and through the point of intersection another line $\parallel$ to the first side, and so on; shew that at the end of the second revolution the line will pass through the original point.

11. If points D, E be taken in the sides BC, CA of a triangle, such that $BD : DC$ as $BA : AE$, and DG be drawn $\parallel$ to CA meeting BE in G ; prove that AG will bisect the angle BAC .

12. D is a point in the side AB of the obtuse-angled triangle ABC having the obtuse angle at A , such that

CD is a fourth proportional to the sides AB , BC , CA ; prove that the triangles ABC , ADC are similar.

13. Describe a circle which shall touch two given straight lines and pass through a given point.

14. To inscribe a square in a given sector of a circle.

15. Given the difference between the diagonal and side of a square; construct the square.

16. Find a point D in the base BC of a triangle ABC such that AD may be a mean proportional between AB and AC .

17. Describe a rhombus equal and equiangular to a given parallelogram.

18. Of all equal and equiangular parallelograms shew that the rhombus has the least perimeter.

19. Produce a straight line which is divided into two parts to a point such that the whole line shall be divided harmonically.

20. Divide a given straight line harmonically.

21. Of all equal rhombuses the square has the least perimeter.

22. Of all equal quadrilateral figures the square has the least perimeter.

23. If two circles touch each other, and also touch a given straight line, the part of the tangent between the points of contact is a mean proportional between the diameters of the circles.

24. In any right-angled triangle one side is to the other as the excess of the hypotenuse above the second

is to the line cut off from the first between the right angle and the line bisecting the opposite angle.

25. Describe a square which shall bear to another a given ratio.

26. In a regular pentagon inscribe a square.

27. Upon a given base describe a triangle having a given vertical angle and one of its sides double of the other.

28. Inscribe in a segment of a circle a rectangle one of whose sides is treble each of the adjacent sides.

29. Divide a triangle into four equal and similar triangles.

30. Describe a triangle similar to a given triangle and having its perimeter equal to a given straight line.

31. The semicircle described on the hypotenuse of a right-angled triangle is equal to the sum of the semicircles described upon the sides.

32. If a straight line be divided into any two parts, and similar figures be described on the whole line and on the parts, then will the perimeter of the former figure be equal to the sum of the perimeters of the latter figures.

33. Two similar rectilineal figures being given, describe another whose perimeter shall be equal to the sum of the perimeters of the two given figures.

34. Two similar rectilineal figures being given, describe another whose perimeter shall be equal to the difference of the perimeters of the two given figures.

35. $ABCD$ is a quadrilateral inscribed in a circle; its diagonals intersect in X : prove that

$$\triangle ABX : \triangle DCX \text{ as sq. on } BX : \text{sq. on } CX.$$

36. If two diagonals of a pentagon be drawn to cut one another, they will each be divided in extreme and mean ratio.

37. APB is a quadrant, of which C is the centre, SPT a straight line touching the arc at P meeting CB , CA produced in S , T , and PM is perpendicular to CA ; prove that

$$\triangle SCT : \triangle ACB \text{ as } \triangle ACB : \triangle CMP.$$

38. From a given point in the side of a triangle to draw straight lines which shall divide the triangle into a given number of equal parts.

39. If lines be drawn from a given point in the circumference of a circle, such that the rectangle between the whole lines and the parts of them within the circle is constant, the locus of their extremities will be a straight line.

40. Given the base, the ratio of the sides containing the vertical angle, and the distance of the vertex from a given point in the base, to construct the triangle.

41. Two similar figures being given, describe a similar figure equal to their sum.

42. Two similar figures being given, describe a similar figure equal to their difference.

43. Describe a figure similar to a given figure, and bearing to it a given ratio.

44. Describe a regular hexagon equal to a given square.
45. Describe a regular hexagon equal to a regular pentagon.
46. Describe a square which shall have the same ratio to a given square which one given straight line has to another.
47. Describe a circle equal to the sum of two given circles.
48. Describe a circle equal to the difference of two given circles.
49. Describe a circle equal to a given quadrant.
50. Describe a circle equal to a given semicircle.
51. If circles be described on the sides of an acute-angled triangle as diameters, any two of them will be together greater than the third.
52. Bisect a circle by a concentric circle.
53. Divide a circle into any given number of equal parts by means of concentric circles.
54. If PS the diameter of a circle be trisected in Q, R , and semicircles be described on PQ, PR on one side of PS , and on QS, RS on the other side of PS , then will the circle be divided into three equal parts having equal perimeters.
55. Divide a circle into any number of equal parts having equal perimeters.

56. If on the bounding radii of a quadrant semicircles be described intersecting one another, then will the area common to the two semicircles be equal to the area intercepted between them and the arc of the quadrant; also the two remaining portions of the semicircles will each be equal to the square on the radius of one of the semicircles.

57. The area of a quadrant is half that of the circle described on its chord; also of the three parts of the circle without the quadrant, the crescent will be equal to the triangular portion of the sector, and the two segments of circles will be together equal to the remaining portion of the sector.

58. The angles of equal sectors are inversely as the squares on their radii.

SOLID GEOMETRY.

1. Trisect a cube.

2. AB is drawn $\perp$ to the plane of the right angle BCD ; prove that CD is $\perp$ to the plane of the angle ABC .

3. If three straight lines are at right angles to one another, then is each straight line $\perp$ to the plane passing through the other two.

4. A perpendicular is drawn from the vertex of a regular tetrahedron on the opposite face, and from its foot a perpendicular is drawn to any one of the other faces; prove that the former line is three times the latter.

5. Find the locus of a point in a plane which is always situated at a given distance from a given point out of that plane.

6. Determine the longest and shortest lines which can be drawn to the circumference of a circle from a given point out of the plane of the circle.

7. From the point of intersection of a straight line with a plane draw a straight line in the given plane making the least possible angle with the given straight line. (This angle is the inclination of the given straight line to the plane.)

8. Determine a plane through a given point perpendicular to a given straight line.

9. If the arms of one angle be $\parallel$ to the arms of another angle, and the straight line joining their vertices is $\perp$ to each of the arms of one angle, then shall it also be $\perp$ to each of the arms of the other angle.

10. If two parallel straight lines meet a given plane, they shall be inclined to it at the same angle.

11. If from a point A above a plane straight lines AB, AC be drawn meeting it in B and C of which AB is perpendicular to the plane, and AC perpendicular to a straight line DC in that plane, and CB joined; CB shall be perpendicular to DC .

12. If three straight lines are mutually $\perp$ to one another, then are the planes passing through them $\perp$ to one another.

13. Draw a straight line $\perp$ to each of two straight lines not in the same plane.

14. Find the shortest path between two straight lines not in the same plane.

15. In any parallelepiped the sum of the squares on the four diagonals is equal to the sum of the squares on the edges.

16. Find the locus of a point equidistant from three given points.

17. AB , BC are $=$ and $\parallel$ to DE , EF respectively, but are not in the same plane with them ; prove that

AD is $=$ and $\parallel$ to CF .

18. If two straight lines intersecting one another be respectively perpendicular to two planes intersecting one another, then will the common section of these planes be perpendicular to the plane through the given straight lines.

19. If the opposite edges of a tetrahedron be equal two and two, prove that the faces are acute-angled triangles.

20. Prove that a tetrahedron can be formed having each of its faces equal in all respects to a given acute-angled triangle.

21. Within the area of a given triangle is described a triangle, the sides of which are parallel to those of the given one. Prove that the sum of the angles subtended by the sides of the interior triangle at any point not in the plane of the triangles is less than the sum of the angles subtended at the same point by the sides of the exterior triangle.

22. The three angles at the vertex of a tetrahedron are together less than the angles subtended by the opposite edges at any point within the solid.

23. The locus of a straight line which is always $\parallel$ to a given straight line and passes through another given straight line is a plane.

24. The locus of a straight line which is always $\perp$ to a given plane and passes through a given straight line in that plane is a plane.

adopted by Aristotle in his peculiar system. The evidence upon obscure or doubtful questions connected with the subject is examined; and the relations which Rhetoric bears, in Aristotle's view, to the kindred art of Logic are considered. A connected Analysis of the treatise is given, sometimes in the form of paraphrase; and a few important matters are separately discussed in Appendices. There is added, as a general Appendix, by way of specimen of the antagonistic system of Isocrates and others, a complete analysis of the treatise called *Ῥητορικὴ πρὸς Ἀλέξανδρον*, with a discussion of its authorship and of the probable results of its teaching.

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